HW assignment 8

Problem

A charge density $\sigma(\theta, \phi) = k \sin\theta \sin\phi$ is glued over the surface of a spherical shell of radius R. Find the resulting potential inside and outside the sphere.

Solution

The general formula for expansion of potential $\phi(r, \theta, \varphi)$ in spherical harmonics has the form

$$\phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (A_{lm} r^l + B_{lm} r^{-l-1}) Y_{lm}(\theta, \varphi)$$

The potential must be continuous at $r = R$ so $B_{lm} = R^{2l+1} A_{lm}$ and therefore the potential can be represented as

$$\phi_{\text{in}}(r, \theta, \varphi) \overset{r \leq R}{=} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} r^l Y_{lm}(\theta, \varphi)$$

$$\phi_{\text{out}}(r, \theta, \varphi) \overset{r \geq R}{=} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} R^{2l+1} r^{-l-1} Y_{lm}(\theta, \varphi)$$

The surface charge density is related to discontinuity in $\vec{E}$

$$E_{\perp}^{\text{above}} - E_{\perp}^{\text{below}} = \frac{\sigma}{\epsilon_0}$$

so we get

$$\frac{\sigma}{\epsilon_0} = \frac{\partial}{\partial r} \phi_{\text{in}}(r, \theta, \varphi) - \frac{\partial}{\partial r} \phi_{\text{out}}(r, \theta, \varphi) \Big|_{r=R}$$

which can be rewritten as

$$\frac{k}{\epsilon_0}\sin\theta\sin\varphi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}R^{l-1}(l+1)Y_{lm}(\theta,\varphi)$$

Using $\sin\theta\sin\varphi = \frac{i}{2}(Y_{1,1} + Y_{1,-1})\sqrt{\frac{8\pi}{3}}$ and the orthonormality of spherical harmonics we get $A_{1,1} = A_{1,-1} = i\frac{k}{6\epsilon_0}\sqrt{\frac{8\pi}{3}}$ and therefore

$$\begin{aligned}\phi_{\text{in}}(r,\theta,\varphi) &= \frac{kr}{3\epsilon_0}\sin\theta\sin\varphi \\ \phi_{\text{out}}(r,\theta,\varphi) &= \frac{kR^3}{3r^2\epsilon_0}\sin\theta\sin\varphi\end{aligned}$$