HW assignment 7. Due Wed Nov 4 at 5 p.m. in my mailbox.

Problem

The potential at the surface of the sphere of radius R is given by

$$V_0(\theta) \;=\; \kappa\cos 3\theta$$

Assuming that there are no charges inside or outside the sphere, find:

(a) the potential inside and outside the sphere,

and

(b) the surface charge density on the sphere.

Solution

(a) The general solution with azimuthal symmetry has the form

$$\phi(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1})P_l(\cos\theta)$$

We need the expansion of $\cos 3\theta$ in Legendre polynomials:

$$\cos 3\theta = \Re(\cos\theta + i\sin\theta)^3 = \cos^3\theta - 3\cos\theta\sin^2\theta = -3\cos\theta + 4\cos^3\theta = \frac{8}{5}P_3(\cos\theta) - \frac{3}{5}\cos\theta$$

where $P_3(\cos\theta) = \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta$. The potential inside the sphere must be regular as $r \to 0$ so

$$\phi_{\rm in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

From the boundary condition

$$\phi(r=R) = \frac{8\kappa}{5}P_3(\cos\theta) - \frac{3\kappa}{5}\cos\theta$$

we see that $A_3 = \frac{8\kappa}{5R^3}$ and $A_1 = -\frac{3\kappa}{5R}$ while all other A_l vanish. Thus, the solution inside the sphere has the form

$$\phi_{\rm in}(r,\theta) = \frac{8\kappa}{5}\frac{r^3}{R^3}P_3(\cos\theta) - \frac{3\kappa}{5}\frac{r}{R}\cos\theta \;=\; 4\kappa\frac{r^3}{R^3}\cos^3\theta - \frac{12\kappa}{5}\frac{r^3}{R^3}\cos\theta - \frac{3}{5}\kappa\frac{r}{R}\cos\theta$$

Similarly, for the solution outside of the sphere we get

$$\phi_{\rm out}(r,\theta) = \frac{8\kappa}{5}\frac{R^4}{r^4}P_3(\cos\theta) - \frac{3\kappa}{5}\frac{R^2}{r^2}\cos\theta \;=\; 4\kappa\frac{R^4}{r^4}\cos^3\theta - \frac{12}{5}\kappa\frac{R^4}{r^4}\cos\theta - \frac{3}{5}\kappa\frac{R^2}{r^2}\cos\theta$$

(b) By Gauss' law

$$\epsilon_0\sigma \;=\; \frac{\partial\phi_{\rm out}(r,\theta)}{\partial r}\Big|_{r=R} - \frac{\partial\phi_{\rm in}(r,\theta)}{\partial r}\Big|_{r=R} \;=\; \frac{\kappa}{5R}\Big(6P_1(\cos\theta) - 32P_3(\cos\theta)\Big) - \frac{\kappa}{5R}\Big(24P_3(\cos\theta) - 3P_1(\cos\theta)\Big)$$

$$= \frac{\kappa}{5R}\Big(9\cos\theta - 56P_3(\cos\theta)\Big) \;\Rightarrow\; \sigma(\theta) \;=\; \frac{\kappa}{\epsilon_0 R}\Big(\frac{93}{5}\cos\theta - 28\cos^3\theta\Big)$$