HW assignment 9 Problem 3.9 from *Jackson.*

Solution. By separation of variables we get

$$\phi(s, \varphi, z) = R(s)T(\varphi)Z(z)$$

where $T(\varphi) = e^{\pm im\varphi}$ and

$$Z(z) = e^{\pm kz}, \qquad R(s) = J_m(ks), \ N_m(ks)$$

or

$$Z(z) = e^{\pm ikz}, \qquad R(s) = I_m(ks), \ K_m(ks)$$

depending on boundary conditions. In our case $\phi(s, \varphi, z)|_{z=0,L} = 0$ so we need the second set with

$$Z(z) = \sin\frac{\pi n z}{L} \quad \Leftrightarrow \quad k_n = \frac{\pi n}{L}$$

Moreover, there are no charges inside the cylinder so the potential must be regular there and therefore $K_m(ks)$ are not acceptable. Thus, our solution has the form

$$\phi(s, \varphi, z) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} c_{mn} I_m(k_n s) \sin k_n z\, e^{im\varphi}$$

To find c_{mn} we take $s = b$ and integrate both sides of this equation over angle φ

$$\int_0^{2\pi} d\varphi\, e^{iM\varphi} V(\varphi, z) = 2\pi \sum_{n=1}^{\infty} c_{Mn} I_M\left(\frac{\pi n b}{L}\right) \sin\frac{\pi n}{L} z \tag{1}$$

and over z using $\int_0^L \sin(\frac{\pi m}{L} z)\sin(\frac{\pi n}{L} z) = \frac{L}{2}\delta_{mn}$

$$\int_0^L dz\, \sin(\frac{\pi N}{L} z) \int_0^{2\pi} d\varphi\, e^{iM\varphi} V(\varphi, z) = \pi L c_{MN} I_M\left(\frac{\pi N b}{L}\right) \tag{2}$$

and therefore

$$c_{mn} = \frac{1}{\pi L I_m(\frac{\pi n b}{L})} \int_0^L dz\, \sin(\frac{\pi n}{L} z) \int_0^{2\pi} d\varphi\, e^{im\varphi} V(\varphi, z)$$