

HW assignment 9

Problem

A charge density $\sigma(\theta, \phi) = k \sin \theta \sin \phi$ is glued over the surface of a spherical shell of radius R . Find the resulting potential inside and outside the sphere.

Solution

The general formula for expansion of potential $\phi(r, \theta, \varphi)$ in spherical harmonics has the form

$$\phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l (A_{lm} r^l + B_{lm} r^{-l-1}) Y_{lm}(\theta, \varphi)$$

The potential must be continuous at $r = R$ so $B_{lm} = R^{2l+1} A_{lm}$ and therefore the potential can be represented as

$$\begin{aligned} \phi_{\text{in}}(r, \theta, \varphi) & \stackrel{r \leq R}{=} \sum_{l=0}^{\infty} \sum_{m=-l}^l A_{lm} r^l Y_{lm}(\theta, \varphi) \\ \phi_{\text{out}}(r, \theta, \varphi) & \stackrel{r \geq R}{=} \sum_{l=0}^{\infty} \sum_{m=-l}^l A_{lm} R^{2l+1} r^{-l-1} Y_{lm}(\theta, \varphi) \end{aligned}$$

The surface charge density is related to discontinuity in \vec{E}

$$E_{\perp}^{\text{above}} - E_{\perp}^{\text{below}} = \frac{\sigma}{\epsilon_0}$$

so we get

$$\frac{\sigma}{\epsilon_0} = \left. \frac{\partial}{\partial r} \phi_{\text{in}}(r, \theta, \varphi) - \frac{\partial}{\partial r} \phi_{\text{out}}(r, \theta, \varphi) \right|_{r=R}$$

which can be rewritten as

$$\frac{k}{\epsilon_0} \sin \theta \sin \varphi = \sum_{l=0}^{\infty} \sum_{m=-l}^l A_{lm} R^{l-1} (l+1) Y_{lm}(\theta, \varphi)$$

Using $\sin \theta \sin \varphi = \frac{i}{2}(Y_{1,1} + Y_{1,-1})\sqrt{\frac{8\pi}{3}}$ and the orthonormality of spherical harmonics we get $A_{1,1} = A_{1,-1} = i\frac{k}{6\epsilon_0}\sqrt{\frac{8\pi}{3}}$ and therefore

$$\begin{aligned} \phi_{\text{in}}(r, \theta, \varphi) &= \frac{kr}{3\epsilon_0} \sin \theta \sin \varphi \\ \phi_{\text{out}}(r, \theta, \varphi) &= \frac{kR^3}{3r^2\epsilon_0} \sin \theta \sin \varphi \end{aligned}$$