604 Midterm (16 points). Tue Nov 26, 10:55 a.m.-12:25 p.m.

Problem 1 (5 points).

An electric dipole p is located a distance d above an infinite grounded conducting plane. If the dipole is free to rotate, in what orientation it will come to rest?

Solution

First, let us calculate the potential energy of the dipole oriented along the θ, ϕ direction in spherical polar coordinates. The mirror image of this dipole is a dipole located a distance d below the $z = 0$ surface and oriented along $\theta, \pi - \phi$ direction. The potential energy is

$$U = \frac{1}{32\pi\epsilon_0 d^3}[(\vec{p}_1 \cdot \vec{p}_2) - 3(\vec{p}_1 \cdot \vec{e}_3)(\vec{p}_2 \cdot \vec{e}_3)] = -\frac{p^2}{4\pi\epsilon_0 d^3}(1 + \cos^2\theta)$$

The dipole will come to rest in $\theta = 0$ or $\theta = \pi$ orientation depending whether the original angle θ was smaller or grater than $\pi/2$.

Problem 2 (5 points).

A free surface charge $\sigma_f = \varsigma \cos\theta$ is glued over the surface of the sphere of radius a made from dielectric with susceptibility χ_e. Find the potential inside and outside of the sphere.

Solution

As usual, when we have asymuthal symmetry we start with

$$\Phi(r,\theta,\varphi) = \begin{cases} \sum_l A_l r^l P_l(\cos\theta) & r < a \\ \sum_l C_l r^{-l-1} P_l(\cos\theta) & r > a \end{cases}, \tag{1}$$

and impose the boundary conditions at the surface of the sphere

$$\begin{aligned} E_\theta(a_-) &= E_\theta(a_+) \quad (\textit{tangential condition}) \\ \epsilon_0 E_r(a_+) - \epsilon E_r(a_-) &= \varsigma \cos\theta \quad (\textit{normal condition}) \end{aligned}$$

Similarly to the example of dielectric sphere in external field (see Eqs. (4.5.13)-(4.5.25) from Chapter 4)

$$E_\theta = \begin{cases} -\sum_l A_l r^{l-1} \frac{d}{d\theta} P_l(\cos\theta) = -\sum_l A_l r^{l-1} P_l^1(\cos\theta) & r < a \\ -\sum_l C_l r^{-l-2} \frac{d}{d\theta} P_l(\cos\theta) = -\sum_l C_l r^{-l-2} P_l^1(\cos\theta) & r > a \end{cases}, \tag{2}$$

and

$$E_r = \begin{cases} -\sum_l A_l \cdot l \cdot r^{l-1} P_l(\cos\theta) & r < a \\ \sum_l C_l (l+1) r^{-l-2} P_l(\cos\theta) & r > a \end{cases}. \tag{3}$$

Using the orthogonality property of the Legendre polynomials we have

$$A_l a^{l-1} = C_l a^{-l-2} \qquad \Rightarrow \qquad A_l = C_l a^{-2l-1}. \tag{4}$$

The normal boundary condition yields

$$\epsilon_0 \sum_l C_l (l+1) a^{-l-2} P_l(\cos\theta) + \epsilon \sum_l A_l \, l \, a^{l-1} P_l(\cos\theta) = \varsigma P_1(\cos\theta) \tag{5}$$

There are two cases

$$\epsilon_0 [C_l (l+1) a^{-l-2}] = -\epsilon A_l l a^{l-1} \quad , \quad l \neq 1 \tag{6}$$

$$2\epsilon_0 C_1 a^{-3} + \epsilon A_1 = \varsigma \quad , \quad l = 1 \tag{7}$$

Substituting Eq. (4) into Eq. (6), we find

$$\epsilon_0[C_l(l+1)a^{-l-2}] = -\,\epsilon C_l a^{-2l-1}\, l a^{l-1} \Rightarrow C_l = 0 \Rightarrow A_l = 0 \quad , \quad l \neq 1. \tag{8}$$

At $l = 1$ from Eq. (4) and Eq. (7) we get

$$A_1 \;=\; \frac{\varsigma}{2\epsilon_0 + \epsilon}, \qquad C_1 \;=\; \frac{\varsigma a^3}{2\epsilon_0 + \epsilon} \tag{9}$$

so the potential is

$$\Phi(r,\theta,\varphi) = \begin{cases} \frac{\varsigma}{2\epsilon_0+\epsilon} r\cos\theta & r < a \\ \frac{\varsigma a^3}{(2\epsilon_0+\epsilon)r^2}\cos\theta & r > a \end{cases} \tag{10}$$

Problem 3 (6 points).

Find the Dirichlet Green function of Laplace equation for the interior of infinite cylinder with radius a.

Solution

Up to Eqs. (3.37) and (3.38) from "Chapter 3" file everything is the same as for infinite space. The difference is in the boundary condition for $y_2(x')$. For infinite space, we had $y_2(x') \to 0$ as $x' \to \infty$ so the proper choice was $y_2(x') = K_m(x')$. Now, the boundary condition is $y_2(ka) = 0$ so we should take

$$y_2(x') \;=\; K_m(x') - \frac{K_m(ka)}{I_m(ka)} I_m(x')$$

The Wronskian $W(y_1(x'), y_2(x') = -\frac{1}{x'}$ is the same as for infinite space case since $W(I(x'), I(x') = 0$ so the Green function can be obtained from Eq. (3.7.45) by replacement of $K_m(ks)$ by

$$L_m(ks) \;=\; K(ks) - \frac{K_m(ka)}{I_m(ka)} I_m(ks)$$

Finally, the Green function reads

$$G(\vec{r}, \vec{r'}) \;=\; \frac{1}{\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk\; e^{ik(z-z')} I_m(|k|s_<) L_m(|k|s_>)$$

A quick check at $a \to \infty$: we get Eq. (3.7.45) since the additional term in L vanishes due to $\frac{K_m(ka)}{I_m(ka)} \overset{a\to\infty}{\to} 0$.