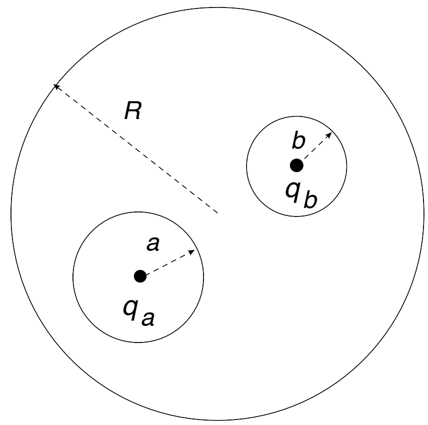
Midterm 1 solutions.

Problem 1.

Two spherical cavities of radii a and b, respectively, are hollowed out from the interior of a neutral metal sphere of radius R as shown below. Point charges q_a and q_b are placed at the centers of the cavities. Find the surface charge densities σ_a and σ_b on the surfaces of the two cavities and the surface charge density σ_R the outer surface of the metal sphere.



Solution

We use superposition principle solve three problems: charge q_a in the cavity hollowed out of an infinite conductor with potential 0, same for charge q_b, and conducting sphere of radius R with charge $q_a + q_b$. The solution of the first problem satisfies Poission equation in the first cavity and has potential $\Phi_1 = 0$ everywhere else, same for the second with $\Phi_2 = 0$ outside the second cavity, and the solution of the third problem satisfies Laplace equation outside the sphere with radius R and has constant potential $\Phi_3 = \frac{q_a+q_b}{4\pi\epsilon_0 R}$ inside that sphere.

The superposition of these three problems has the potential

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3 \qquad (*)$$

which is constant ($=\Phi_3$) throughout our conductor with two cavities. In addition, the potential inside the first cavity is

$$\Phi = \frac{q_a}{4\pi\epsilon_0|\vec{r} - \vec{r}_a|} + \Phi_3$$

which obviously satisfies Poisson equation

$$\nabla^2\Phi = -\frac{q_a}{\epsilon_0}\delta(\vec{r} - \vec{r}_a)$$

since Φ_3=const there. Similarly, the potential inside the second sphere satisfies correct Poisson equation, and the potential outside the sphere of radius R satisfies Laplace equation. Thus, our superposition $(*)$ satisfies correct Poisson equations and boundary conditions that the potential throughout the conductor is constant.

Finally, the surface charge on the first cavity is the same as in auxiliiary problem

$$\sigma_a = -\frac{q_a}{4\pi a^2}$$

and similarly

$$\sigma_b = -\frac{q_b}{4\pi b^2}$$

As seen from the third auxiliary problem, the charge on the outer surface of the sphere is distributed symmetrically so

$$\sigma_R \;=\; \frac{q_a + q_b}{4\pi R^2}$$

Problem 2.

A conducting sphere of radius R_1 carries charge Q. A second, initially uncharged conducting sphere of radius R_2 is placed at large distance $R \gg R_1, R_2$ and then connected to the first sphere by a very thin wire with large resistance. After a long time, the system of two conducting spheres reaches equilibrium.

(a)
Find the electrostatic force between two spheres

(b)
Find the ohmic heat dissipated in the wire and the spheres. Neglect effects of radiation.

Solution

(a)
The potential of a charged conducting sphere of radius a is

$$\Phi \;=\; \frac{q}{4\pi\epsilon_0 a}$$

At the equilibrium, the potentials of two spheres must be the same so

$$\frac{Q_1}{4\pi\epsilon_0 R_1} = \frac{Q - Q_1}{4\pi\epsilon_0 R_2} \quad \Rightarrow Q_1 = Q\frac{R_1}{R_1 + R_2}, \quad Q_2 = Q\frac{R_2}{R_1 + R_2}$$

Force between two charges is

$$F \;=\; \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \;=\; \frac{Q^2}{4\pi\epsilon_0 R^2}\frac{R_1 R_2}{(R_1 + R_2)^2}$$

(b)
In the beginning, the energy of the first conducting sphere is

$$E \;=\; \frac{Q^2}{8\pi\epsilon_0 R_1}$$

In the end the energies of the spheres are

$$E_1 \;=\; \frac{Q_1^2}{8\pi\epsilon_0 R_1}, \qquad E_2 \;=\; \frac{Q_2^2}{8\pi\epsilon_0 R_2}$$

At the absence of radiation, the difference of energies is dissipated as the ohmic heat so

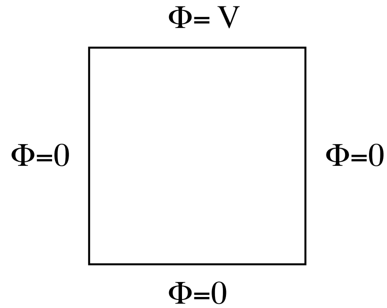
$$\text{Heat} \;=\; \frac{Q^2}{8\pi\epsilon_0 R_1} - \frac{Q^2}{8\pi\epsilon_0}\frac{R_1}{(R_1 + R_2)^2} - \frac{Q^2}{8\pi\epsilon_0}\frac{R_1}{(R_1 + R_2)^2} \;=\; \frac{Q^2}{8\pi\epsilon_0}\frac{R_2}{(R_1 + R_2)^2}$$

Problem 3 (5 points).

Find the solution of Laplace equation in a 2-dimensional square well of size a where three sides are kept at zero potential and the fourth side at constant potential V.

Solution
By analogy with the problem from Lecture 9

$$\Phi(x, y) \;=\; \sum_{n=1}^{\infty} A_n \sinh \alpha_n y \sin \alpha_n x, \qquad \alpha_n = \frac{\pi n}{a}$$



Boundary conditions at $x = 0, a$ and $y = 0$ are trivially satisfied. At $y = a$ we get

$$V = \sum_{n=1}^{\infty} A_n \sinh \alpha_n a \sin \alpha_n x$$

so

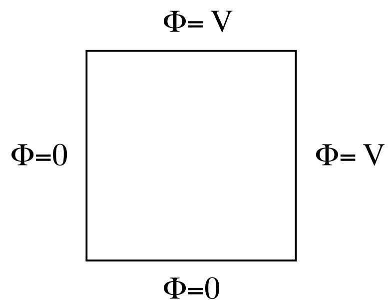
$$A_n = \frac{2V}{a \sinh \pi n} \int_0^a \sin \frac{\pi n x}{a} = \frac{2V}{a \sinh \pi n} \left[1 - (-1)^n\right]$$

and the solution takes the form

$$\Phi(x, y) = \frac{4V}{\pi} \sum_{n=\mathrm{odd}} \frac{1}{n \sinh \pi n} \sin \frac{\pi n x}{a} \sinh \frac{\pi n y}{a}$$

Extra credit - 3 points.

Same for setup shown below



Solution
By superposition principle

$$\Phi(x, y) \;=\; \frac{4V}{\pi} \sum_{n=\text{odd}} \frac{1}{n \sinh \pi n} \sin \frac{\pi n x}{a} \sinh \frac{\pi n y}{a} \;+\; (x \leftrightarrow y)$$