604 Midterm (16 points).

Problem 1.

A surface charge glued over the (non-conducting) sphere of radius R produces the potential

$$\phi(x,y,z) \;=\; \frac{V_0}{2R^2}(2z^2-x^2-y^2)$$

inside the sphere. Find the potential outside the sphere and the surface charge density.

Solution
Expand the potential in Legendre polynomials. The potential inside is

$$\phi^{\text{in}}(r,\theta) \;=\; \frac{V_0 r^2}{2R^2}(3\cos^2\theta-1)$$

The potential is continuous so outside of the sphere we have

$$\phi^{\text{out}}(r,\theta) \;=\; \frac{V_0 R^3}{2r^3}(3\cos^2\theta-1)$$

The surface charge density is

$$\sigma(r,\theta) \;=\; \epsilon_0(E^{\text{out}}-E^{\text{in}})\big|_{r=R} \;=\; \epsilon_0\Big(\frac{\partial\phi^{\text{in}}}{\partial r}-\frac{\partial\phi^{\text{out}}}{\partial r}\Big)\big|_{r=R} \;=\; \frac{5\epsilon_0 V_0}{2R}(3\cos^2\theta-1)$$

Problem 2.

Find the Dirichlet Green function of Laplace equation for the interior of infinite cylinder with radius a.

Solution
Up to Eqs. (3.37) and (3.38) from "Chapter 3" file everything is the same as for infinite space. The difference is in the boundary condition for $y_2(x')$. For infinite space, we had $y_2(x')\to 0$ as $x'\to\infty$ so the proper choice was $y_2(x')=K_m(x')$. Now, the boundary condition is $y_2(ka)=0$ so we should take

$$y_2(x') \;=\; K_m(x')-\frac{K_m(ka)}{I_m(ka)}I_m(x')$$

The Wronskian $W(y_1(x'),y_2(x')=-\frac{1}{x'}$ is the same as for infinite space case since $W(I(x'),I(x')=0$ so the Green function can be obtained from Eq. (3.7.45) by replacement of $K_m(ks)$ by

$$L_m(ks) \;=\; K(ks)-\frac{K_m(ka)}{I_m(ka)}I_m(ks)$$

Finally, the Green function reads

$$G(\vec r,\vec r') \;=\; \frac{1}{\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}dk\; e^{ik(z-z')}I_m(|k|s_<)L_m(|k|s_>)$$

A quick check at $a\to\infty$: we get Eq. (3.7.45) since the additional term in L vanishes due to $\frac{K_m(ka)}{I_m(ka)}\overset{a\to\infty}{\to}0$.

Problem 3.

A half-space below $z=0$ is filled with dielectric with susceptibility $\chi_e>1$. A pure dipole $\vec p=p\hat e_1$ is located at distance d above the dielectric. Find the force acting on the dipole. Is it attractive or repulsive?

Solution
The image dipole is $\vec p'=-p\frac{\chi_e-1}{\chi_e+1}$. A general formula for the force acting on the dipole $\vec p$ located at $\vec x$ due to the dipole $\vec p'$ located at $\vec x'$ is

$$\vec F(\vec x) \;=\; \frac{1}{4\pi\epsilon_0}\vec\nabla_x\Big\{\frac{3(\vec p'\cdot(\vec x-\vec x'))(\vec p\cdot(\vec x-\vec x'))-(\vec p\cdot\vec p')|\vec x-\vec x'|^2}{|\vec x-\vec x'|^5}\Big\}$$

In our case this reduces to

$$\vec{F}(\vec{x}) \;=\; -\frac{1}{4\pi\epsilon_0}(\vec{p}\cdot\vec{p}')\vec{\nabla}_x\frac{1}{|\vec{x}-\vec{x}'|^3} \;=\; -\frac{3p^2}{4\pi\epsilon_0}\frac{\chi_e-1}{\chi_e+1}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^5} \;=\; -\frac{3p^2}{64d^4\pi\epsilon_0}\frac{\chi_e-1}{\chi_e+1}\hat{e}_3$$

Thus, the force is attractive, the direction is down, and the magnitude is $\frac{3p^2}{4\pi\epsilon_0}\frac{\chi_e-1}{\chi_e+1}(2d)^{-4}$

All problems have equal weight.