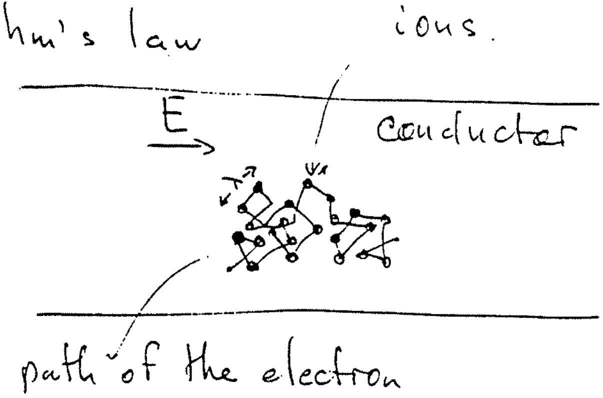
Electrodynamics

1. Ohm's law



$|v_{thermal}| = const$
direction is random

electron bounces from the ions ⇒
$\Rightarrow t = \frac{\lambda}{v_{thermal}}$ $\quad \lambda$ - average distance between ions

$v_{average} = \frac{1}{2}at = \frac{a\lambda}{2v_{ther}}$

$\vec{a} = \frac{\vec{F}}{m_e} = \frac{e}{m_e}(\vec{E} + \cancel{\vec{v}\times\vec{B}})$ - for $v \ll c$

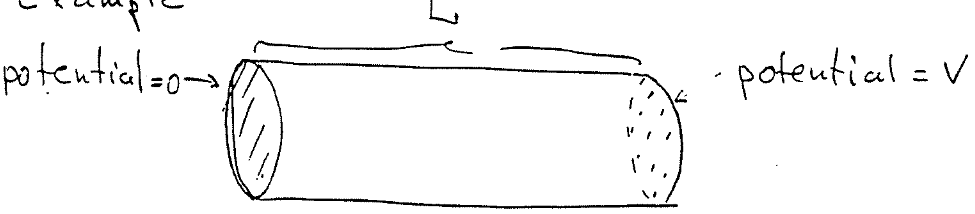
If we have N molecules per unit volume and f free electrons per molecule,

$$\vec{J} = Nf\,\vec{v}_{ave} = \underbrace{\left(\frac{fN\lambda e^2}{2m_e v_{thermal}}\right)}_{\sigma}\vec{E}$$

σ - „conductivity"
$\rho \equiv 1/\sigma$ - „resistivity"

$$\boxed{\vec{J} = \sigma\vec{E}}$$
Ohm's law

Example



The field inside the cylinder is uniform (may be proved)
$\Rightarrow E = \frac{V}{L}$ ↑↑ axis of the cylinder $\Rightarrow J = \sigma\frac{V}{L}$

$\Rightarrow I = J\underset{\uparrow}{A} = \sigma A\, V/L$
area of the cross section

$V = \underbrace{\frac{L}{\sigma A}}_{R} I$
R - „resistance"

$V = IR$ - traditional form of Ohm's law

Work done by the electrical force is converted into heat in the resister

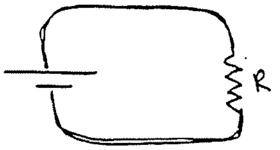
$$\left.\begin{array}{l}\Delta W = \Delta e\, V \\ \Delta e = I\Delta t\end{array}\right\} \Rightarrow \Delta W = VI\Delta t \Rightarrow P = \frac{\Delta W}{\Delta t} = VI = I^2R$$

„Joule heating law"

Electromotive force (emf).

Ohm'law : $\vec{J} = \sigma \vec{f}$

↖ force exerted per unit charge



$$\vec{f} = \vec{f}_s + \vec{E}$$

↑ source

↖ electrostatic force which smoothen out the flow

Definition

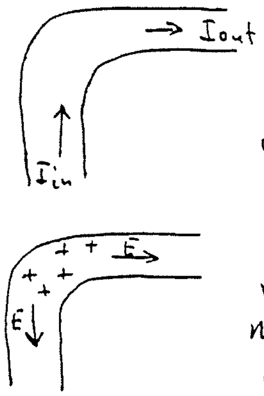
$$\mathcal{E} \equiv \oint \vec{f} \cdot d\vec{l}$$ „electromotive force"

Since $\oint \vec{E} \cdot d\vec{l} = 0 \Rightarrow$

$\Rightarrow \mathcal{E} = \oint \vec{f}_s \cdot d\vec{l}$

⇓

$\mathcal{E}$ can be interpreted as the work done by the source per unit charge

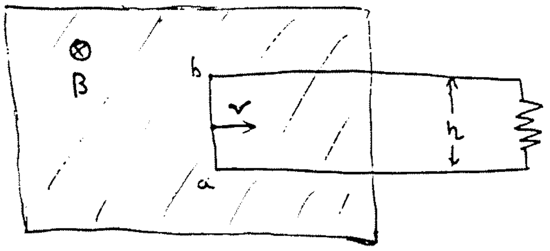


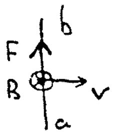
$I_{in} \neq I_{out} \Rightarrow$
$\Rightarrow$ there will be accumulation of ⊕ charges in the bend $\Rightarrow$ they will produce electric field which will correct the mismatch between I_{in} and I_{out}

$$\mathcal{E} = \oint \vec{f} \cdot d\vec{l} = \oint \frac{\vec{J}}{\sigma} \cdot d\vec{l} = \oint \frac{I}{a\sigma} dl = I \oint \frac{1}{a\sigma} dl = IR$$ $\Rightarrow$ $\mathcal{E}$ determine current in the circuit.

Motional emf.

Elementary generator:





$\vec{F}_{mag} = e\vec{v} \times \vec{B} \Rightarrow$

$\Rightarrow \vec{f}_{mag} = \vec{v} \times \vec{B}$

$|f_{mag}| = vB$

$$\Rightarrow \mathcal{E} = \oint \vec{f}_{mag} \cdot d\vec{l} = vBh$$

Power converted into the heat is $P = I^2 R$ (where $I = \frac{\mathcal{E}}{R}$)

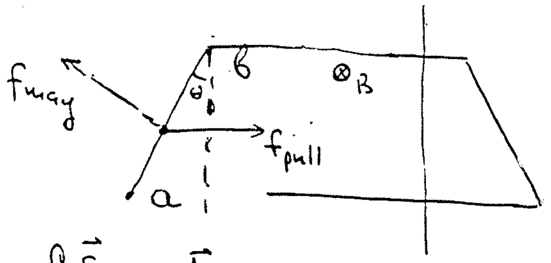
Who is doing this job (≡ this work)?

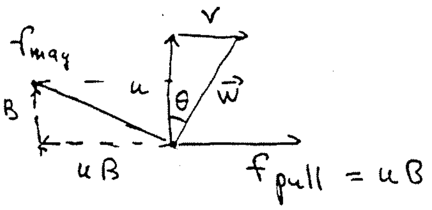
Wrong A; magnetic force, because it is responsible for $\mathcal{E}$.

↖ magnetic forces <u>never</u> do work!

Right A: force exerted by the person who pulls the wire

Actual path of the unit of charge:





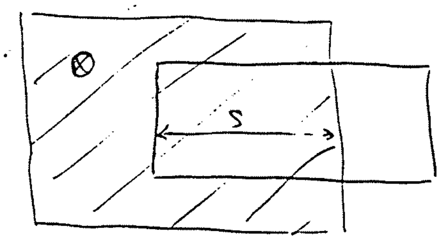
$$l_{ab} = \frac{h}{\cos\theta}$$

$$\oint \vec{f}_{pull}\cdot d\vec{l} = (uB)\left(\frac{h}{\cos\theta}\right)\cos\left(\frac{\pi}{2}-\theta\right) = uBh\,tg\,\theta = vBh$$

⇒ work done on unit of charge is equal to emf.

However, this work is done by f_{pull}, since f_{mag} does no work. On the other hand, f_{pull} contributes nothing to emf since it is ⊥ to the wire.

Flux rule

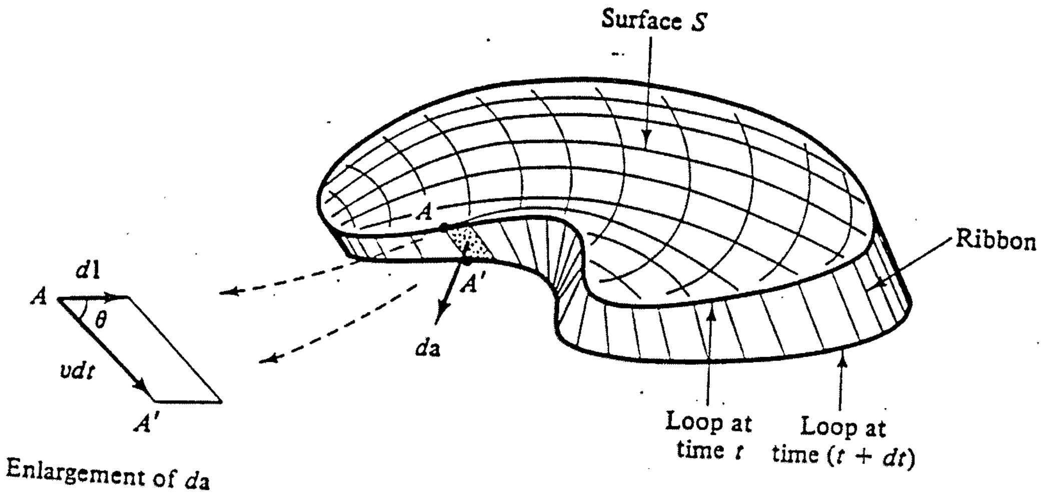


$$\Phi = \int \vec{B}\cdot d\vec{a} = Bhs$$

$$\frac{d\Phi}{dt} = Bh\frac{ds}{dt} = -Bhv \quad \Rightarrow \quad \boxed{\mathcal{E} = -\frac{d\Phi}{dt}} \quad \leftarrow \text{flux rule for motional emf}$$

This is a general rule for any loop moving in a static magnetic field

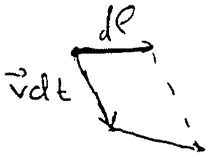
Proof:



Enlargement of da

Change of the flux = flux through the ribbon

$$d\Phi = \int_{\text{ribbon}} \vec{B}\cdot d\vec{a}$$



$$d\vec{a} \simeq (\vec{v}\times d\vec{\ell}\,)\,dt$$

$$\Rightarrow \frac{d\Phi}{dt} = \oint \vec{B}\cdot(\vec{v}\times d\vec{\ell}\,)$$

Let us introduce velocity of the charge

$\vec{w} = \vec{v} + \vec{u}$

where $\vec{u}$ is the velocity of the charge down the wire

$\vec{u} = \frac{d\vec{\ell}}{dt}$

Since $\vec{u}\uparrow\uparrow d\vec{\ell}$ $\quad \vec{v}\times d\vec{\ell} = (\vec{v}+\vec{u})\times d\vec{\ell} = \vec{w}\times d\vec{\ell}$

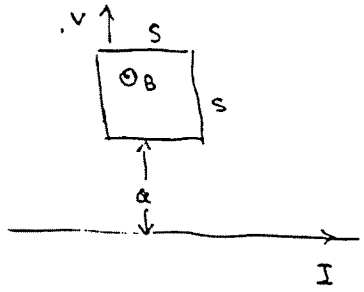
$$\Rightarrow \frac{d\Phi}{dt} = \oint \vec{B}\cdot(\vec{w}\times d\vec{\ell}\,) = -\oint \underbrace{(\vec{w}\times\vec{B})}_{\vec{f}_{mag}}\cdot d\vec{\ell}$$

$\vec{f}_{mag}$ ≡ magnetic ~~force~~ per unit charge

$$\Rightarrow \frac{d\Phi}{dt} = -\oint \vec{f}_{mag}\cdot d\vec{\ell} = -\mathcal{E}$$

$$\Rightarrow \mathcal{E} = -\frac{d\Phi}{dt}$$

Example:



$$\text{Flux} = \int \vec{B}\cdot d\vec{a} = s\int_a^{a+s} B(r)\,dr = s\int_a^{a+s} \frac{\mu_0 I}{2\pi r}\,dr =$$

$$= \frac{\mu_0}{2\pi} I s \ln\frac{a+s}{a}$$

If we pull up with speed v

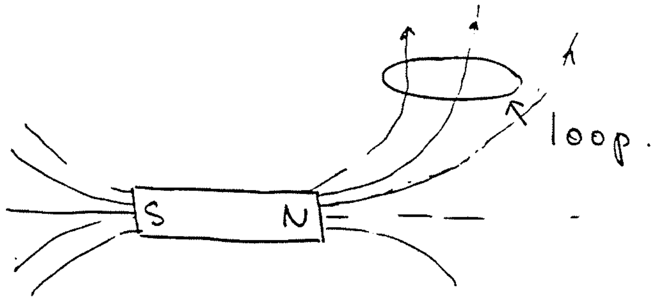
$a \to a_0 + vt$

$$\Rightarrow \phi = \frac{\mu_0}{2\pi} I s \ln\frac{a_0+vt+s}{a_0+vt} \Rightarrow$$

$$\Rightarrow \mathcal{E} = -\frac{d\phi}{dt} = -\frac{\mu_0}{2\pi} I s\left(\frac{v}{a_0+vt+s} - \frac{v}{a_0+vt}\right) = -\frac{\mu_0}{2\pi} I s^2 v\,\frac{1}{a(a+s)}$$

If we pull to the right, no change of flux ⇒ no emf.

Faraday's law



1. We move the loop. E.m.f. is due to magnetic force.
$\mathcal{E} = -\frac{d\phi}{dt}$

2. We move the magnet. The result should be the same (relativity), but if the loop does not move, there are no magnetic forces to produce the e.m.f.
What happens: changing magnetic field produces an electric field which gives us the nessesary e.m.f.

$\oint \vec{E} \cdot d\vec{\ell} = \mathcal{E} = -\frac{d\phi}{dt}$ „Faraday's law in integral form"

$\Downarrow$
Stokes' theorem
$\Downarrow$

$$\oint \vec{E} \cdot d\vec{\ell} = \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{a} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} \Rightarrow$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ „Faraday's law in differential form"

This is one of the four fundamental Maxwell's equations for electrodynamics

Lenz law: how to remember signs in Faraday's law
The em.f. produces such current that the magnetic field due to this current tends to counteract the change of flux that induced the e.m.f.

Lenz law applies to motional e.m.f., too.

How to calculate the induced electric field

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \xleftrightarrow[\text{to}]{\text{similar}} \qquad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla}\cdot\vec{E} = 0 \quad \text{(no charges)} \qquad\qquad \vec{\nabla}\cdot\vec{B} = 0.$$

But we know the solution to the second pair of equations:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\tau'\, \vec{J}(r') \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

$\Rightarrow$ the solution of the first pair of equations is obtained by the replacement $\mu_0 \vec{J} \longrightarrow -\frac{\partial \vec{B}}{\partial t}$:

$$\vec{E}(\vec{r}) = -\frac{1}{4\pi}\int d\tau'\, \frac{\partial \vec{B}(r')}{\partial t} \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = \frac{d}{dt}\Big\{ \underbrace{-\frac{1}{4\pi}\int d\tau'\, \vec{B}(\vec{r}\,') \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}}_{\vec{A}(\vec{r})} \Big\}$$

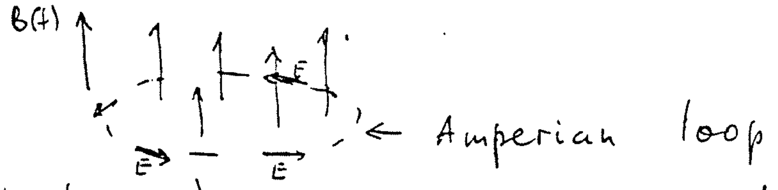
$$\Rightarrow \quad \vec{E}(\vec{r}) = -\frac{\partial \vec{A}(\vec{r},t)}{\partial t}$$

Check: $\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = -\frac{\partial \vec{B}}{\partial t}$

In the integral form

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc} \qquad \xleftrightarrow[\text{to}]{\text{similar}} \qquad \oint \vec{E}\cdot d\vec{\ell} = -\frac{d\phi}{dt}.$$

Example:
Time-dependent uniform magnetic field

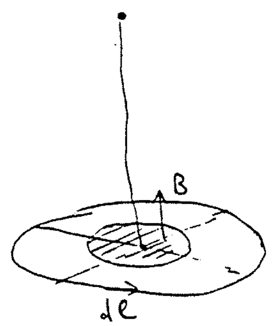


Situation here is similar to the magnetic field of the uniform current in $\vec{e}_3$ direction $\Rightarrow$
$\Rightarrow$ induced electric field is circumferential

$$\oint \vec{E}\cdot d\vec{\ell} = E\, 2\pi r = -\frac{d\phi}{dt} = -\frac{d}{dt}\pi r^2 B = -\pi r^2 \frac{dB}{dt} \qquad \Rightarrow \quad E = -\frac{r}{2}\frac{dB}{dt}$$

If B_0 is decreasing, $\vec{E}$ is counterclockwise $\Rightarrow$ it tries to induce current which will support the decreasing flux.

Another example



Charged rim is suspended so that it is free to rotate. (Line charge density is λ)

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\phi}{dt} = -\pi a^2 \frac{dB}{dt}$$

Torque on segment $d\vec{\ell}$ is

$\vec{R} \times d\vec{F}$ $\quad d\vec{F} = \lambda d\ell \cdot \vec{E} \Rightarrow$

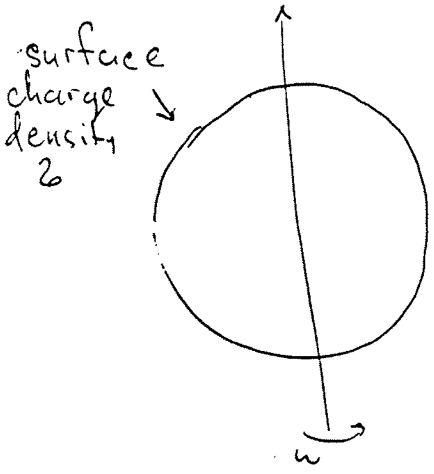
$\Rightarrow d\vec{N} = \vec{R} \times \vec{E}\, \lambda d\ell \rightarrow d\vec{N} = RE\lambda d\ell\, \vec{e}_3$

The total torque is

$N = R\lambda \oint E\, d\ell = -R\lambda \pi a^2 \frac{\partial B}{\partial t}$ and the direction is $\uparrow\uparrow \vec{e}_3$

NB: Quasistatic approximation: we use Faraday's law for changing magnetic fields but use magnetostatics (Bio-Savart law etc) to calculate these magnetic fields. It is OK in most practical cases (unless we have the strong radiation of electromagnetic waves at very high frequencies).

Example: spinning spherical shell with time-dependent angular velocity



$\vec{E} = \vec{E}_c + \vec{E}_f$

$\vec{E}_c$ = ordinary Coulomb field =

$= \frac{1}{4\pi\epsilon_0} \frac{Q}{r^3} \vec{r}\, \theta(r-R)$

$\vec{E}_f$ = Faraday field due to changing magnetic field =

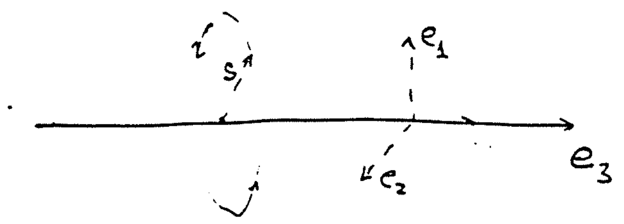
$= -\frac{\partial \vec{A}}{\partial t}$

$$\vec{A} = \frac{\mu_0 R \sigma}{3} \omega(t)\, r \sin\theta\, \hat{\varphi}\, \theta(R-r) + \frac{\mu_0 R^4 \sigma}{3} \omega(t) \frac{\sin\theta}{r^2} \hat{\varphi}\, \theta(r-R) \Rightarrow$$

$$\Rightarrow \vec{E}_f = -\frac{\mu_0 R \sigma}{3} \frac{d\omega}{dt} r \sin\theta\, \hat{\varphi}\, \theta(R-r) - \frac{\mu_0 R^4 \sigma}{3} \frac{d\omega}{dt} \frac{\sin\theta}{r^2} \hat{\varphi}\, \theta(r-R)$$

Provided that $\omega(t)$ changes slowly

Example : induced electric field due to the infinite straight wire with current I(t)



In quasistatic approximation $\vec{B} = \frac{\mu_0}{2\pi} I(t)\,\hat{\varphi} \Rightarrow$

$$\Rightarrow \vec{A}(s,t) = -\frac{\mu_0 I(t)}{2\pi} \ln s\, \vec{e}_3$$

$$\Rightarrow \vec{E}(s,t) = -\frac{\mu_0}{2\pi} \frac{dI(t)}{dt} \ln s\, \vec{e}_3$$

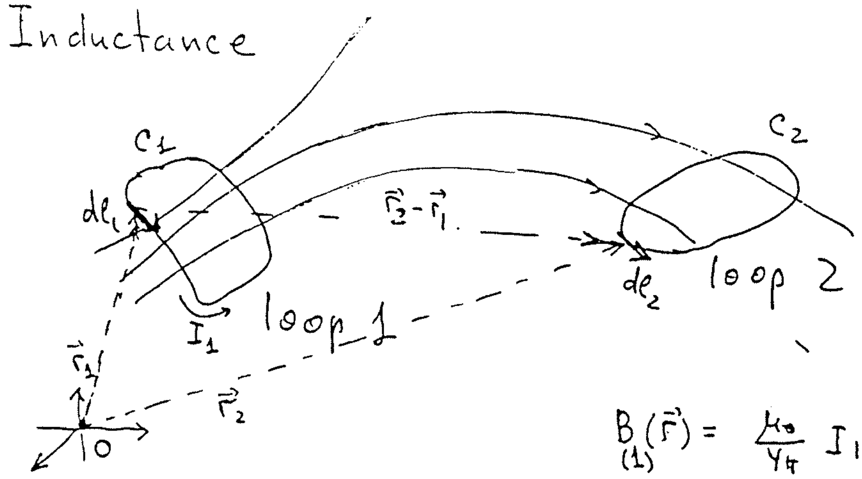
Actually, it is a little bit more tricky since the correct formula for vector potential of a long straight wire of length L was

$$\vec{A} = \frac{\mu_0}{2\pi} I \vec{e}_3 \ln L/s$$

$$\Rightarrow \vec{E}(s,t) = \frac{\mu_0}{2\pi} \frac{dI(t)}{dt} \ln \frac{L}{s}\, \vec{e}_3 = \frac{\mu_0}{2\pi} \frac{dI(t)}{dt} (\ln L - \ln s)\, \vec{e}_3$$

The term $\frac{\mu_0}{2\pi} I \ln L\, \vec{e}_3$ was an unobservable additional constant term in the vector potential $A(\vec{s}\,)$ in the case of time-independent current. Now it is perfectly observable electric field so one must be careful while using $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$. A better way is to solve the equation $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ anew (with appropriate boundary condition $\vec{E} \to 0$ at infinity)

Inductance



$$B_{(1)}(\vec r) = \frac{\mu_0}{4\pi} I_1 \oint d\vec l_1 \times \frac{(\vec r - \vec r_1)}{|\vec r - \vec r_1|^3}$$

⇒ $B_{(1)}$ is proportional to I_1 ⇒ the flux of $B_{(1)}$ through loop 2 is also proportional to I_1

$\Phi_2 = M_{21} I_1$ M_{21} – „mutual inductance" of the two loops.

Why „mutual"

$$\Phi_2 = \int \vec B_{(1)} \cdot da_2 = \int (\vec\nabla \times \vec A_{(1)}) \cdot d\vec a_2 \overset{\text{Stokes' theorem}}{=} \oint_{C_2} \vec A_{(1)} d\vec l_2$$

$$\left.\vec A_{(1)}(\vec r) = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} d\vec l_1 \frac{1}{|\vec r - \vec r_1|} \Rightarrow \vec A_{(1)}(r_2) = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} d\vec l_1 \frac{1}{|\vec r_2 - \vec r_1|} \right\} \Rightarrow$$

$$\Rightarrow \Phi_2 = \oint_{C_2} \frac{\mu_0 I_1}{4\pi} \oint_{C_1} d\vec l_1 \frac{1}{|\vec r_2 - \vec r_1|} \cdot d\vec l_2 = \frac{\mu_0}{4\pi} I_1 \oint_{C_1} \oint_{C_2} \frac{d\vec l_1 \cdot d\vec l_2}{|\vec r_1 - \vec r_2|}$$

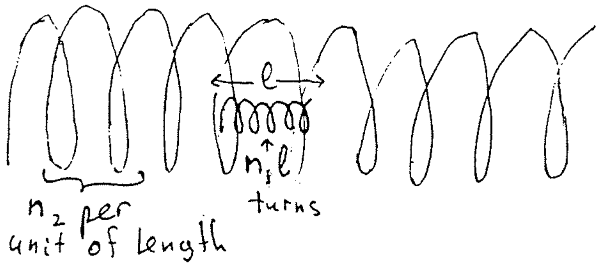
$$\Rightarrow M_{21} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec l_1 \cdot d\vec l_2}{|\vec r_1 - \vec r_2|}$$ „Neumann formula"

This double integral is rather difficult for practical purposes, but two important properties are seen immediately:

1. M_{21} is purely geometrical quantity
2. $M_{21} = M_{12}$ ⇒ we will call it M – the mutual inductance of the two loops

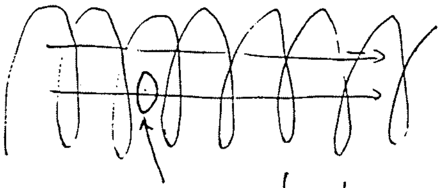
Corollary of (2): For any two loops the flux through loop 2 when we run a current I in the loop 1 is the same as the flux through loop 1 when we run the same current I around loop 2.

Example : small solenoid inside the large one.



Suppose we run a current I in the short solenoid. What will be the flux through the long one?

To solve this problem by direct computation of the flux through the large solenoid is technically impossible. On the other hand, to calculate the flux through small solenoid when I is flowing in the large one is very simple



$B = \mu_0 n_2 I$

a single turn from small solenoid. The flux is $B \cdot \pi R^2$ (R is the radius of small solenoid)

$\Rightarrow$ Flux through one turn $= \mu_0 n_2 I \pi R^2$

Flux through $N_1 \ell$ turns $= \mu_0 n_1 n_2 \ell I \pi R^2 =$

$\Rightarrow \Phi = \mu_0 n_1 n_2 \ell I \pi R^2$ is also the flux through large solenoid when we run current I in the small one

$$M_{12} = \mu_0 n_1 n_2 \pi R^2 \ell.$$

If we vary current in loop 1

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}$$ – e.m.f. in the loop 2.

But the flux through loop 1 also varies $\Rightarrow$ there will be an e.m.f. in the loop 1 itself

$$\mathcal{E}_1 = -\frac{d\Phi_1}{dt} \qquad \Phi_1 = LI$$

„self-inductance"

$$\Rightarrow \mathcal{E} = -L\frac{dI}{dt}$$

Self-inductance L is also determined by the geometry of the loop.

Example: self-inductance of a solenoid

$B_{\text{inside}} = \mu_0 \ \ I$

↓

number of turns per unit length

Flux thru 1 loop

$\int B\cdot da = B\pi a^2$

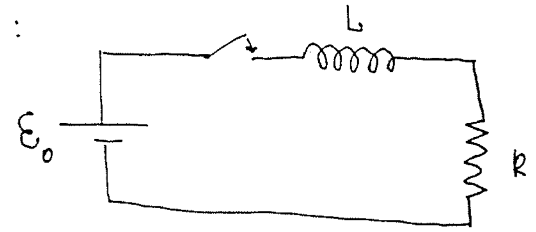
Total flux: $NB\pi a^2$ $N = n\ell$

$\Phi = n\ell B\pi a^2 \Rightarrow$

$\Phi = \mu_0 n^2 I \pi a^2 \ell \Rightarrow M = \mu_0 n^2 \pi a^2 \ell$

Due to the Lenz law, (self-) inductance is always positive

Example:



What happens if we turn on the switch

Ohm's law $\mathcal{E}_0 + (-L\frac{dI}{dt}) = IR$

↑ e.m.f. due to self-inductance

$\Rightarrow L\frac{dI}{dt} = \mathcal{E}_0 - IR$ ← first-order differential equation

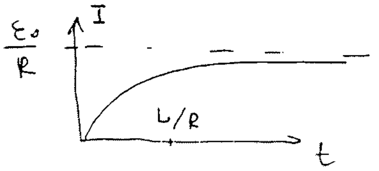
$$\frac{dI}{dt} = \frac{\mathcal{E}_0}{L} - I\frac{R}{L} \Rightarrow \frac{dI}{\mathcal{E}_0/L - I/L\,R} = dt \Rightarrow \int \frac{dI}{\mathcal{E}_0/L - I/L\,R} = \frac{t}{L}R \Rightarrow$$

$$\Rightarrow \ln\left(\frac{\mathcal{E}_0}{L} - \frac{I}{L}R\right) = -\frac{tR}{L} + c \Rightarrow -\frac{I}{L}R + \frac{\mathcal{E}_0}{L} = e^c e^{-\frac{tR}{L}} \Rightarrow I = \frac{\mathcal{E}_0}{R} - ke^{-t\frac{R}{L}}$$

$(k = e^c \frac{L}{R})$

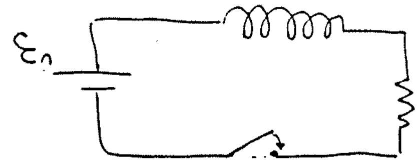
the constant k is to be determined from initial condition $I(t)|_{t=0} = 0 \Rightarrow k = \frac{\mathcal{E}_0}{R}$

$\Rightarrow I(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-t\frac{R}{L}}\right)$



$\frac{L}{R}$ = „time constant"

Energy in magnetic fields



What is the energy stored in the inductance L?

Work done on a unit charge against the back emf. in one trip around the circuit is $-\mathcal{E}$. The amount of charge per unit of time is $I \Rightarrow$

work per unit of time $= -\mathcal{E}I = LI\frac{dI}{dt} \Rightarrow \frac{dw}{dt} = LI\frac{dI}{dt}$

The total work is

$$W = \int_0^\infty \frac{dw}{dt}\,dt = \int_0^\infty LI\frac{dI}{dt}\,dt = L\int_0^I I\,dI = \frac{LI^2}{2}$$ ← energy stored in the inductan

(This formula is similar to $w = \frac{1}{2}cV^2$ - electrostatic energy stored in the capacitor with capacitance c).

One can rewrite $W = \frac{LI^2}{2}$ in a very nice way

$$LI = \Phi = \int_S \vec{B}\cdot d\vec{a} = \int_S (\vec{\nabla}\times\vec{A})\cdot da = \oint_C \vec{A}\cdot d\vec{l}$$

c - boundary of the surface s

vector potential

$$\Rightarrow W = \frac{1}{2}I\oint_C \vec{A}\cdot d\vec{l} = \frac{1}{2}\oint_C (\vec{A}\cdot\vec{I})\,dl$$

For the volume currents we must replace $\vec{I}\,dl$ by $\vec{J}\,d\tau$

$$\Rightarrow W = \frac{1}{2}\int_{\text{volume}} (\vec{A}\cdot\vec{J})\,d\tau = \frac{1}{2\mu_0}\int_{\text{Volume}} \vec{A}\cdot(\vec{\nabla}\times\vec{B})\,d\tau$$

$$A\cdot(\vec{\nabla}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}^2 - \vec{\nabla}\cdot(\vec{A}\cdot$$

$$\Rightarrow W = \frac{1}{2\mu_0}\int_{\text{volume}} d\tau\, B^2 - \underbrace{\frac{1}{2\mu_0}\int_{\text{volume}} d\tau\, \nabla\cdot(\vec{A}\times\vec{B})}$$

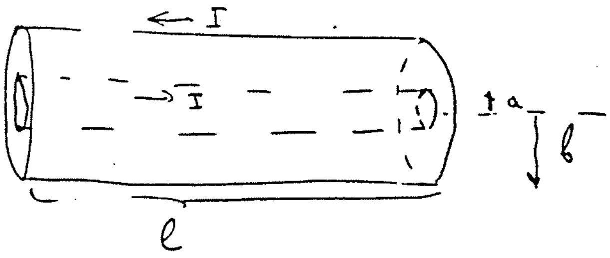
$$\frac{1}{2\mu_0}\int_{\text{Surface}} (\vec{A}\times\vec{B})\cdot d\vec{a} = 0$$ if the surface lies on infinity and the field decreases

$$\Rightarrow W = \frac{1}{2\mu_0}\int_{\text{all space}} d\tau\, B^2$$ ← „energy stored in the magnetic field"

The total energy stored in the electromagnetic field is

$$W = \frac{1}{2}\int_{\text{all space}} d\tau\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$$

Example : coaxial cable



Q : Find the energy stored in a section of length l

Between the cylinders $\vec{B}(r) = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$.
Elsewhere, $\vec{B} = 0$

$$\Rightarrow W = \frac{1}{2\mu_0}\int\limits_{\substack{\text{between}\\ \text{cylinders}}} d\tau\, B^2 = \frac{l}{2\mu_0}\int_a^l dr\,d\varphi\, r\, B^2 = \frac{l\pi}{\mu_0}\int_a^l dr\, r\, B^2 =$$

$$= \frac{l\pi}{\mu_0}\int_a^b dr\, r\,\frac{\mu_0^2 I^2}{4\pi^2 r^2} = \frac{\mu_0 I^2}{4\pi} l \int_a^l \frac{dr}{r} = \frac{\mu_0 I^2}{4\pi} l \ln\frac{l}{a}$$

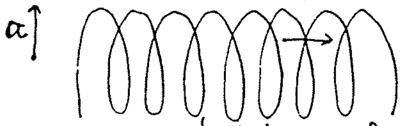
$$\Rightarrow W = \frac{\mu_0 I^2}{4\pi} l \ln\frac{l}{a}$$

Comparing this to the formula $W = \frac{1}{2}LI^2$ we can find the inductance :

$$L = \frac{\mu_0}{2\pi} l \ln\frac{b}{a}$$

Example : solenoid

inside : $\vec{B} = \mu_0 n I \hat{\varphi}$



outside : $B = \varnothing$

$$\Phi_{\text{one loop}} = B\pi a^2 = \mu_0 n I \pi a^2 \Rightarrow$$

$$\Rightarrow \Phi = \mu_0 n^2 I \pi a^2 l \Rightarrow L = \mu_0 \pi a^2 n^2 l$$

1) $W = \frac{1}{2}LI^2 \quad L = \mu_0 \pi a^2 n^2 l \Rightarrow W = \frac{1}{2}\mu_0 \pi a^2 n^2 I^2 l$

2) $W = \int\limits_{\text{all space}} \frac{B^2}{2\mu_0} d^3x' = \frac{1}{2\mu_0}\int\limits_{\text{cylinder}} B^2 d^3x' = \frac{1}{2\mu_0}(\mu_0 n I)^2 \underbrace{\int\limits_{\text{cylinder}} d^3x'}_{\pi a^2 l} = \frac{\mu_0}{2}\pi a^2 l n^2 I^2$

Maxwell's equations.

Before Maxwell

$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$ Gauss law

$\vec{\nabla} \cdot \vec{B} = 0$

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faraday's law

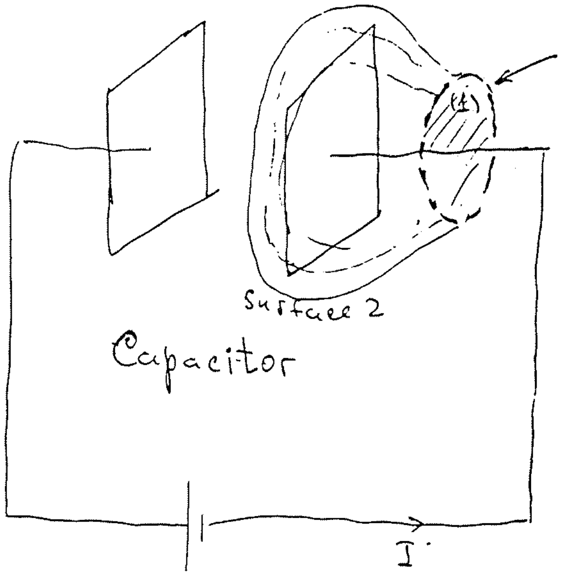
$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ Ampere's law

Maxwell realized that there is a problem with Ampere's law if one wants to apply it outside magnetostatics.

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 (\vec{\nabla} \cdot \vec{J})$. On the other hand,

$\vec{\nabla} \cdot (\vec{\nabla} \times \overrightarrow{\text{whatever}}) = 0$ → contradiction if $\vec{\nabla} \cdot \vec{J} \neq 0$ (for non-steady currents)

Less formally: consider an example



Ampere's law in the integral form reads:

$\oint \vec{B} \cdot d\vec{\ell} = I_{enc} \mu_0$

But if we draw surface 1 (plane circle)

$I_{enc} = I$

whereas if we draw surface 2

$I_{enc} = 0$ ⟹

For non-steady currents the current enclosed by the loop is an ill-defined notion.

How to fix this problem

Maxwell's solution : suppose we have

$$\vec\nabla\times\vec B = \mu_0\vec J + (\overrightarrow{\text{smth else}}),$$

then

$$\vec\nabla\cdot(\mu_0\vec J) + \vec\nabla\cdot(\overrightarrow{\text{smth else}}) = 0 \Rightarrow$$

$$\Rightarrow \vec\nabla\cdot(\overrightarrow{\text{smth else}}) = -\vec\nabla\cdot\vec J\mu_0$$

But, due to continuity equation (conservation of charges

$$\vec\nabla\cdot\vec J = -\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0\vec\nabla\cdot\vec E) = -\vec\nabla\cdot\left(\epsilon_0\frac{\partial\vec E}{\partial t}\right) \Rightarrow$$

$$\Rightarrow \vec\nabla(\overrightarrow{\text{smth else}}) = \vec\nabla\left(\epsilon_0\frac{\partial\vec E}{\partial t}\right)$$

the first guess that comes to mind is that $\overrightarrow{\text{smth else}} = \epsilon_0\frac{\partial\vec E}{\partial t}$, and experiments confirm that it is true

$$\Rightarrow \quad \vec\nabla\times\vec B = \mu_0\vec J + \underbrace{\mu_0\epsilon_0}_{=\frac{1}{c^2}}\frac{\partial\vec E}{\partial t}$$

· Ampere's law with Maxwell's correction

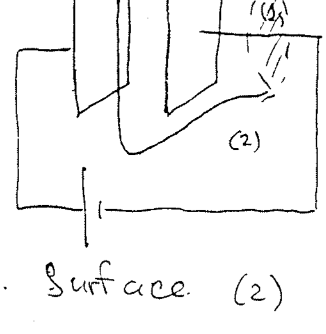
this new term is very small numerically (essentially, it is a relativistic effect since $\mu_0\epsilon_0 = 1/c^2$) and this is the reason that it was never observed by Faraday and others.

Maxwell's form

$$\vec\nabla\times\vec B = \mu_0(\vec J + \vec J_d) \qquad \vec J_d = \epsilon_0\frac{\partial\vec E}{\partial t}$$

„displacement current"

Let us check again the example with the capacitor



$$\oint\vec B\cdot d\vec l = \mu_0 I_{enc} + \mu_0\epsilon_0\int\frac{\partial\vec E}{\partial t}\cdot d\vec a$$

1. Surface (1) $\vec E = 0$ outside the capacitor $\Rightarrow \frac{\partial E}{\partial t} = 0 \Rightarrow$

$$\Rightarrow \oint\vec B\cdot d\vec l = \mu_0 I_{enc} = \mu_0 I$$

2. Surface (2) $E_{inside} = \frac{\sigma}{\epsilon_0} \Rightarrow \left(\frac{\partial E}{\partial t}\right)_{inside} = \frac{1}{\epsilon_0}\frac{\partial\sigma}{\partial t} = \frac{1}{\epsilon_0}\frac{1}{A}\underbrace{\frac{dQ}{dt}}_{I(t)}$

$$\Rightarrow \oint\vec B\cdot d\vec l = \underbrace{\cancel{\mu_0 I_{enc}}}_{=0} + \frac{\mu_0\epsilon_0}{A}\int\frac{1}{\epsilon_0}I\,da = \mu_0 I \Rightarrow \text{same result}$$

Four Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

(*) $$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{B}}{\partial t}$$

They are almost symmetrical with respect to $\vec{E}$ and $\vec{B}$ If there were the magnetic monopoles, the equations would be

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\mu_0\vec{L} - \frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B} = \mu_0\eta \qquad \nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

η - „density" of magnetic monopoles and L is their „curren

We do not see them in the experiment ⇒ probably the do not exist ~~for~~ some unknown reason.