Solution to the beam problem

(1)

Choose the z axis in the direction of velocity of electron beam. In the lab frame

$$\rho \;=\; -\,ne, \qquad I \;=\; \rho v A \;=\; -\,nevA$$

The frame K_0 moves with velocity $\vec{v} = v\hat{z}$ so in the rest rame K_0 of electrons the density is

$$\rho^{(0)} \;=\; \gamma_v\big(\rho - \frac{v}{c^2}J_z\big) \;=\; \gamma_v\big(ne - \frac{v}{c^2}nev\big) \;=\; \frac{ne}{\gamma_v},$$

Check of the current in K_0: $J_z^{(0)} \;=\; J_z - v\rho \;=\; 0$ as expected.

(2)

With respect to frame K_0, positron is moving with velocity

$$-u \;=\; -\,\frac{v+v}{1+\frac{v^2}{c^2}} \;=\; -\,\frac{2v}{1+\frac{v^2}{c^2}}$$

Reciprocally, electron beam is moving with velocity

$$u \;=\; \frac{2v}{1+\frac{v^2}{c^2}}$$

with respect to positron's frame K'. The density of electrons in K' frame is

$$\rho' \;=\; \gamma_u \rho^{(0)} \;=\; -\,\gamma_u\frac{ne}{\gamma_v} \;=\; -\,ne\gamma_v\big(1+\frac{v^2}{c^2}\big)$$

where $\gamma_u = \frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} \;=\; \big(1+\frac{v^2}{c^2}\big)\gamma_v^2$.

The electric field due to the beam can be obtained from the Gauss law. In cylindrical coordinates

$$\vec{E}' \;=\; \frac{\lambda/\epsilon_0}{2\pi s}\hat{s} \;=\; \frac{\rho' A}{2\pi\epsilon_0 s}\hat{s} \;=\; -\,\gamma_v\big(1+\frac{v^2}{c^2}\big)\frac{neA}{2\pi\epsilon_0 s}\hat{s}$$

The magnetic field in K' frame can be obtained from the Ampere law

$$\vec{B}' \;=\; \mu_0\frac{I}{2\pi s}\hat{\phi} \;=\; \mu_0\frac{\rho' u}{2\pi s}\hat{\phi} \;=\; -\,\mu_0\gamma_v v\frac{neA}{\pi s}\hat{\phi}$$

The Lorentz force acting on the positron in K' frame is

$$\vec{F}' \;=\; e\vec{E}' \;=\; -\,\gamma_v\frac{neA}{2\pi\epsilon_0 s}\hat{s}\big(1+\frac{v^2}{c^2}\big)\hat{s}$$

Since momentum in the transverse $\hat{s}$ direction does not change and since $t = \gamma_v\tau$, transforming force to K frame we get

$$F_s \;=\; \frac{dp_s}{dt} \;=\; \frac{1}{\gamma_v}\frac{dp_s}{d\tau} \;=\; \frac{1}{\gamma_v}F_s' \;=\; -\,\frac{neA}{2\pi\epsilon_0 s}\hat{s}\big(1+\frac{v^2}{c^2}\big)$$

(3)

In lab K frame

$$\vec{E} \;=\; \frac{\rho A}{2\pi\epsilon_0 s}\hat{s} \;=\; -\,\frac{neA}{2\pi\epsilon_0 s}\hat{s}, \qquad \vec{B} \;=\; -\,\mu_0 v\frac{neA}{2\pi s}\hat{\phi}$$

so the Lorentz force is $(\hat{z}\times\hat{\phi} = -\hat{s})$

$$\vec{F} \;=\; e(\vec{E}+\vec{v}\times\vec{B}) \;=\; e(\vec{E} - v\vec{z}\times\vec{B}) \;=\; \frac{neA}{2\pi\epsilon_0 s}\hat{s}(1+\mu_0\epsilon_0 v^2) \;=\; -\,\frac{neA}{2\pi\epsilon_0 s}\hat{s}\big(1+\frac{v^2}{c^2}\big)$$

in accordance with part (2).

Check of Lorentz transformation

$$E_s' \;=\; \gamma_v\big(E_s - v(\hat{z}\times\vec{B})\cdot\hat{s}\big) \;=\; -\,\gamma_v\Big(\frac{neA}{2\pi\epsilon_0 s} - \mu_0 v^2\frac{neA}{2\pi s}(\hat{z}\times\hat{\phi})\cdot\hat{s}\Big) \;=\; -\,\gamma_v\frac{neA}{2\pi\epsilon_0 s}\hat{s}\big(1+\frac{v^2}{c^2}\big)$$

$$B_\phi' \;=\; \gamma_v\big(B_\phi + \frac{v}{c^2}(\hat{z}\times\vec{E})\cdot\hat{\phi}\big) \;=\; -\,\gamma_v\Big(\mu_0 v\frac{neA}{2\pi s} + \frac{v}{c^2}\frac{neA}{2\pi\epsilon_0 s}(\hat{z}\times\hat{s})\cdot\hat{\phi}\Big) \;=\; -\,\mu_0\gamma_v\frac{neA}{2\pi s}2v$$