Contents

Chapter 11

Special Theory of Relativity and Covariant Electrodynamics

11.1 Lorentz Transformations

Central to Newtonian Mechanics is the concept of an **inertial frame**; a frame in which a body, acted on by no external forces, moves with a constant velocity. A transformation between two inertial frames is known as a **Galilean Transformation**.
Aside: a practical definition of an inertial frame is one moving with constant velocity relative to the distant stars (Mach's principle).

11.1.1 Galilean Transformations

Consider two inertial frames K, K', moving with a relative constant velocity $\mathbf{v}$. The coordinates in the two frames are related by

$$\begin{aligned} t' &= t \,, \\ \mathbf{x}' &= \mathbf{x} - \mathbf{v}t \,. \end{aligned} \tag{11.1.1}$$

Consider the interactions of an ensemble of N particles at positions $\mathbf{x}_i; i = 1, \dots, N$, acting solely under the influence of a central potential $V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|)$. Then the equation of motion of particle i in K is

$$m_i \frac{d\mathbf{v}_i}{dt} = -\sum_j \nabla_{x_i} V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|).$$

Suppose that we look at the equation of motion in K'. Then we should have

$$m_i \frac{d\mathbf{v'}_i}{dt} = -\sum_j \nabla_{x'_i} V_{ij}(|\mathbf{x}'_i - \mathbf{x}'_j|).$$

It is evident that $\mathbf{v}'_i = \mathbf{v}_i - \mathbf{v}$, and under Eq. (11.1.1),

$$\frac{\partial}{\partial x'_i} = \frac{\partial}{\partial x_i} \; .$$

We also have $d\mathbf{v}'{}_i/dt = d\mathbf{v}_i/dt$ and

$$|\mathbf{x}'_i - \mathbf{x}'_j| = |\mathbf{x}_i - \mathbf{x}_j| \; .$$

Thus, we see that the equation of motion in K' is of exactly the same form as that in K – we say that classical Newtonian mechanics transforms **covariantly** under Galilean Transformations.

11.1.2 Maxwellian Mechanics under Galilean Transformations

We have seen that electric and magnetic propagation in a vacuum satisfies the wave equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(x,y,z;t) = 0. \tag{11.1.2}$$

Let us now consider the transformation of this equation under Eq. (11.1.1). We have

$$\begin{aligned}
\frac{\partial}{\partial x_i} &= \frac{\partial x'_j}{\partial x_i}\frac{\partial}{\partial x'_j} + \frac{\partial t'}{\partial x_i}\frac{\partial}{\partial t'} \\
&= \delta_{ij}\frac{\partial}{\partial x'_j} + 0 = \frac{\partial}{\partial x'_i} \\
\frac{\partial}{\partial t} &= \frac{\partial x'_j}{\partial t}\frac{\partial}{\partial x'_j} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} \\
&= -v_i\frac{\partial}{\partial x'_i} + \frac{\partial}{\partial t'} = \frac{\partial}{\partial t'} - \mathbf{v}\cdot\nabla' \; .
\end{aligned}$$

Thus the wave equation (11.1.2) becomes

$$\begin{aligned}
\left[\nabla'^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial t'} - \mathbf{v}\cdot\nabla'\right)\left(\frac{\partial}{\partial t'} - \mathbf{v}\cdot\nabla'\right)\right]\psi &= 0 \\
\text{i.e.} \quad \left[\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} + \frac{2}{c^2}\mathbf{v}\cdot\nabla'\frac{\partial}{\partial t'} - \frac{1}{c^2}(\mathbf{v}\cdot\nabla')(\mathbf{v}\cdot\nabla')\right]\psi &= 0 \qquad (11.1.3)
\end{aligned}$$

This equation is clearly different from equation (11.1.2). *The wave equation does not transform covariantly under Galilean Transformations.* For sound waves there is no problem; they propagate in a medium, and it is natural to formulate the wave equation in a frame in which the medium is at rest. Thus the natural question arose - *Is there a frame in which the "ether" is at rest"?*. Of course, we all know the answer (Michelson-Morley) that the velocity of light is the same in all frames, and the resolution of this nasty transformation property is the **Special Theory of Relativity**.

11.1.3 Postulates of Special Theory of Relativity

1. The same laws of nature hold in all inertial systems moving uniformly with respect to one another.

2. The velocity of light has the same value in all systems moving uniformly with respect to each other, independent of velocity of observer relative to the source.

11.1.4 Lorentz Transformations and Kinematic Results of Special Relativity

We will now derive the relationship between coordinates in two frames K, K' moving with constant velocity $\mathbf{v}$ relative to one another. Without a loss of generality, we will let the origin of the coordinates coincide at $t = t' = 0$.
We suppose that a flashlight is rapidly switched on and off at the origin at $t = t' = 0$. Then, by postulate 2, observers in both K and K' see a spherical shell of radiation expanding with the velocity of light c. The wavefront satisfies

$$\begin{aligned} \text{In } K: &\qquad c^2t^2 - (x^2 + y^2 + z^2) = 0 \\ \text{In } K': &\qquad c^2t'^2 - (x'^2 + y'^2 + z'^2) = 0 \end{aligned}$$

Thus we see that, under such a transformation, the quantity $c^2t^2 - (x^2 + y^2 + z^2) = 0$ remains invariant. The emission of the light, and its subsequent absorption at some later times, are each **events**. We have considered the case where the events are separated by something traveling at the speed of light. More generally, we can define the combination

$$\boxed{\Delta s^2 = c^2\Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2)\ , \quad (11.1.4)}$$

where $\Delta t = t_2 - t_1$, $\Delta x = x_2 - x_1$, etc., the **interval** between two events (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2). As we shall see, it is invariant under transformations between inertial frames. To derive the form of the transformation keeping invariant the combination in Eq. (11.1.4), we will specialize to the case where the axes in K, K' are parallel, and the frames are moving with a relative velocity $\mathbf{v} = v\mathbf{e_3}$. Because the transformation must reduce to the Galilean transformation in the limit of small relative velocities, we need consider only the linear relations

$$\begin{aligned} t' &= a_1 t + b_1 z \\ z' &= a_2 t + b_2 z \\ x' &= x \\ y' &= y \end{aligned} \tag{11.1.5}$$

The transverse dimensions do not change (see the gedanken experiment of Taylor and Wheeler discussed in *Griffith's* textbook).
Because the frames are moving with relative velocity v, we have that the event $z' = 0$ corresponds to $z = vt$, yielding

$$a_2 = -vb_2.$$

We now impose invariance of Δs^2:

$$c^2t^2 - (x^2 + y^2 + z^2) = c^2(a_1 t + b_1 z)^2 - x^2 - y^2 - (a_2 t + b_2 z)^2 \ ,$$

which we can expand as

$$c^2t^2[1 - a_1^2 + a_2^2/c^2] - z^2[1 + b_1^2 c^2 - b_2^2] + 2zt[a_2 b_2 - c^2 a_1 b_1] = 0.$$

This is true $\forall x, t$, so equating the coefficients to zero yields

$$\begin{aligned} a_1^2 - a_2^2/c^2 &= 1 \\ b_2^2 - c^2 b_1^2 &= 1 \\ a_2 b_2 &= c^2 a_1 b_1. \end{aligned}$$

Using $a_2 = -vb_2$ converts the system into

$$\begin{aligned} a_1^2 - b_2^2 v^2/c^2 &= 1 \\ b_2^2 - c^2 b_1^2 &= 1 \\ b_2^2 &= -c^2 a_1 b_1/v \ . \end{aligned}$$

Excluding b_2^2, we have

$$\begin{aligned} a_1^2 + a_1 b_1 v &= 1 \\ -c^2 a_1 b_1/v - c^2 b_1^2 &= 1 \ . \end{aligned}$$

Substituting

$$b_1 = (1 - a_1^2)/a_1 v$$

into the second equation produces

$$-\frac{c^2}{v^2}(1-a_1^2) - \frac{c^2}{v^2 a_1^2}(1-a_1^2)^2 \;\; = \;\; 1$$

or

$$(1-a_1^2) + \frac{1}{a_1^2}(1-a_1^2)^2 \;\; = \;\; -\frac{v^2}{c^2} \; ,$$

which simplifies into

$$\frac{1}{a_1^2} - 1 \;\; = \;\; -\frac{v^2}{c^2} \; .$$

Thus,

$$a_1^2 \;\; = \;\; \frac{1}{1 - v^2/c^2} \equiv \gamma^2 \; .$$

The *gamma-factor*

$$\gamma \;\; \equiv \;\; \frac{1}{\sqrt{1 - v^2/c^2}} \; .$$

plays important role in the coordinate transformations of special relativity.
Note, that for zero velocity v we have $\gamma^2 = 1$ and hence $a_1^2 = 1$. Since a_1 relates t' at the origin $z = 0$ to t, choosing positive

$$a_1 = +\gamma$$

would mean that t' runs in the same direction as t, i.e. there is no *time inversion*. For the $b_1 = (1/a_1^2 - 1)a_1/v$ coefficient this gives

$$b_1 = -\frac{v}{c^2}\,\gamma \; ,$$

and hence

$$ct' = \gamma \left[ct - \frac{v}{c} z \right] \; .$$

Now,

$$b_2^2 = -c^2 a_1 b_1 / v = \gamma^2 \; .$$

Choosing

$$b_2 = +\gamma$$

for the coefficient b_2 relating z' to z at the initial moment of time $t = 0$ means that there is no *z-axis inversion* involved in our coordinate transformation. Finally, we have $a_2 = -vb_2$, or

$$a_2 = -v\gamma$$

which gives

$$z' = \gamma\left[z - \frac{v}{c}ct\right] .$$

For completeness, we also recall the relations

$$\begin{aligned} x' &= x , \\ y' &= y \end{aligned}$$

between the remaining coordinates.
We can write these transformations in an axis-independent form as

$$\left.\begin{aligned} ct' &= \gamma(ct - \beta x_\parallel) \\ x'_\parallel &= \gamma(x_\parallel - \beta ct) \\ \mathbf{x}'_\perp &= \mathbf{x}_\perp \end{aligned}\right\} \tag{11.1.6}$$

(check: $c^2t'^2 - {x'_\parallel}^2 = \gamma^2(c^2t^2 - x_\parallel^2) + \gamma^2\beta^2(x_\parallel^2 - c^2t^2) = \gamma^2(1-\beta^2)(c^2t^2 - x_\parallel^2) = c^2t^2 - x_\parallel^2$)
where

$$\begin{aligned} \beta &= v/c , \\ \gamma &= (1-\beta^2)^{-1/2} , \\ \mathbf{x}_\parallel &= \frac{\mathbf{x}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{\boldsymbol{\beta}\cdot\mathbf{x}}{\beta} . \end{aligned} \tag{11.1.7}$$

In vector form, this is

$$\begin{aligned} ct' &= \gamma(ct - \boldsymbol{\beta}\cdot\mathbf{x}) \\ \mathbf{x}' &= \mathbf{x} + \frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}ct . \end{aligned} \tag{11.1.8}$$

It is easy to derive the inverse transformation

$$\left.\begin{aligned} ct &= \gamma(ct' + \beta x'_\parallel) \\ x_\parallel &= \gamma(x'_\parallel + \beta ct) \end{aligned}\right\} \tag{11.1.9}$$

Note, that it involves $-\beta$, in accordance with the fact that K moves with respect to K' with the opposite velocity $-v$.

11.1.5 Rapidity

Let us introduce a parameter ζ, called *rapidity*, defined by

$$\beta \equiv \tanh\zeta = \frac{\sinh\zeta}{\cosh\zeta} .$$

When ζ changes from 0 to ∞, β changes from 0 to 1. An inverse transformation may be found from

$$\beta = \frac{e^{\zeta} - e^{-\zeta}}{e^{\zeta} + e^{-\zeta}} \quad \Rightarrow \quad e^{2\zeta} = \frac{1+\beta}{1-\beta} \tag{11.1.10}$$

or

$$\zeta = \frac{1}{2} \ln \left(\frac{1+\beta}{1-\beta} \right) \ . \tag{11.1.11}$$

We also have

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{\cosh\zeta}{\sqrt{\cosh^2\zeta - \sinh^2\zeta}} = \cosh\zeta$$

and

$$\beta\gamma = \tanh\zeta \cosh\zeta = \sinh\zeta \ .$$

Then, for frames moving parallel to the z axis, we have

$$\begin{aligned} ct' &= ct\cosh\zeta - z\sinh\zeta \\ z' &= z\cosh\zeta - ct\sinh\zeta, \end{aligned} \tag{11.1.12}$$

which has the form of a "rotation" by a complex angle $\phi = i\zeta$, namely

$$\begin{aligned} (ict') &= (ict)\cos\phi - z\sin\phi \\ z' &= z\cos\phi + (ict)\sin\phi \ , \end{aligned} \tag{11.1.13}$$

or

$$\begin{aligned} x_4' &= x_4\cos\phi - z\sin\phi \\ z' &= x_4\sin\phi + z\cos\phi \ , \end{aligned} \tag{11.1.14}$$

with $x_4 \equiv ict$ being the imaginary "fourth" coordinate. The "Euclidean" rotation (11.1.14) in the (x_4, z) plane does not change the value of $x_4^2 + z^2 = -(c^2t^2 - z^2)$, i.e. the interval between the event (x_4, t) and the $t = 0$ event at the origin $z = 0$. Moreover, such a rotation does not change the interval

$$\Delta s^2 = -(x_4^{(2)} - x_4^{(1)})^2 - (z^{(2)} - z^{(1)})^2 = c^2\Delta t^2 - \Delta z^2 \tag{11.1.15}$$

between any two arbitrary events $(x_4^{(1)}, z^{(1)})$ and $(x_4^{(2)}, z^{(2)})$ on the (x_4, z) plane.

11.1.6 Kinematical Properties of Lorentz Transformations

Given two events $(ct_1, \mathbf{x}_1)$ and $(ct_2, \mathbf{x}_2)$, Lorentz transformations leave the **interval**

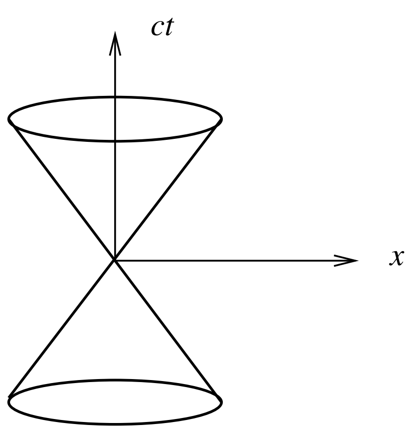
$$\Delta s^2 = c^2(t_2 - t_1)^2 - (\mathbf{x}_2 - \mathbf{x}_1)^2$$

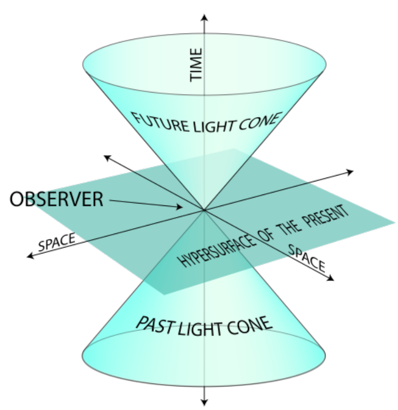
invariant. Thus we can classify the interval by the **sign** of Δs^2, as follows

- $\Delta s^2 > 0$. This is **timelike** separation. We have $c|t_2 - t_1| > |\mathbf{x}_2 - \mathbf{x}_1|$, so that the two points can communicate by a signal traveling at *less than* the speed of light, and indeed a frame can be chosen such that $|\mathbf{x}_2 - \mathbf{x}_1| = 0$.
- $\Delta s^2 = 0$. This is **lightlike** separation. We have $c|t_2 - t_1| = |\mathbf{x}_2 - \mathbf{x}_1|$, so that the two points can only be connected by a signal traveling *at* the speed of light.
- $\Delta s^2 < 0$. This is **spacelike** separation, with $c|t_2 - t_1| < |\mathbf{x}_1 - \mathbf{x}_2|$. The two space-time points cannot communicate, and indeed a frame exists in which $t_1 = t_2$.

11.1.7 Light Cone

Points that can be connected with the space-time origin by a light signal are said to lie on the **light cone**.

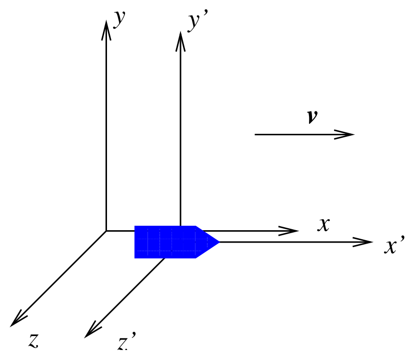




Points within the light cone can be causally connected with the origin, whilst those outside cannot. The forward $(ct > 0)$ and backward $(t < 0)$ cones define absolute future and absolute past, and the ordering is preserved under Lorentz transformations.

11.1.8 Simultaneity, Length Contraction and Time Dilation

Consider a rocket moving with constant velocity v along the x direction relative to the lab frame K. Let us denote the rest frame of the rocket by K'. We assume that the axes of the frames are parallel, and the origins coincide at $t = 0$.
On the side of a rocket is a meter rule. We also have, in the lab. frame, a high density of observers, each with a very accurate clock synchronized in the frame K.



Simultaneity

At time t, an observer in the lab frame, co-incident with one end of the meter rod, records his position $(ct, \mathbf{x}_1)$, and an observer coincident with the other end does likewise $(ct, \mathbf{x}_2)$. Thus $(ct, \mathbf{x}_1)$ and $(ct, \mathbf{x}_2)$ denote two events, which are *simultaneous* in the lab. frame.
In the rocket rest frame K' we have

$$\begin{aligned} ct'_1 &= \gamma(ct - \beta x_1) \\ x'_1 &= \gamma(x_1 - \beta ct) \\ ct'_2 &= \gamma(ct - \beta x_2) \\ x'_2 &= \gamma(x_2 - \beta ct) \end{aligned} \tag{11.1.16}$$

We immediately see that $t'_1 = t'_2$ iff $x_1 = x_2$; in general the points are *not simultaneous* in the rocket rest frame.

Length Contraction

In the rocket frame, our meter rule has length $x'_2 - x'_1$. However, from Eq. (11.1.16), we see that in the laboratory frame the length is obtained from

$$x'_2 - x'_1 = \gamma(x_2 - x_1),$$

i.e.

$$\boxed{x_2 - x_1 = \frac{x_2' - x_1'}{\gamma}}$$

Since $\gamma \geq 1$, we have that length is **contracted**: in a frame, in which the meter rule is moving (lab frame), its length is smaller than in the frame where the meter rule is at rest (rocket frame).

Time Dilation

We now imagine that the clocks in K, K' are synchronized at $t_1 = t_1' = 0$ as the rocket passes origin in frame K. An observer at some point x_2 in K records the time t_2 at which rocket passes x_2, and an observer in K' records time t_2' at which he passes the laboratory observer. The rocket observer is always at $x_2' = 0$, so we have

$$\begin{aligned} 0 &= \gamma(x_2 - \beta c t_2) \\ \Longrightarrow x_2 &= \beta c t_2 \end{aligned}$$

From the third equation of (11.1.16), $ct_2' = \gamma(ct_2 - \beta x_2)$, we have

$$ct_2' = \gamma(ct_2 - \beta x_2) = \gamma(ct_2 - \beta^2 ct_2) = \gamma\, ct_2\,(1 - \beta^2) = ct_2/\gamma\ ,$$

or

$$\boxed{t_2' = \frac{t_2}{\gamma}.}$$

Thus we see that time is **dilated**: a clock that is at rest (lab frame) shows a larger time between two events than a moving clock (rocket frame) .

11.1.9 Proper Time

We now generalize the discussion to the case where the rocket is moving with a velocity $\mathbf{v}(t)$ along some path relative to the lab frame K. We will now introduce K' as the **instantaneous rest frame** of the rocket.

Consider two closely separated points on the trajectory, with coordinates in the two frames $\{(ct, \mathbf{x}), (c[t+dt], \mathbf{x} + d\mathbf{x})\}$ and $\{(ct', \mathbf{x}'), (c[t'+dt'], \mathbf{x}' + d\mathbf{x}')\}$ respectively.

The interval between the points is the invariant, and we have

$$ds^2 = c^2 dt'^2 - \mathbf{dx}'^2 = c^2 dt^2 - \mathbf{dx}^2.$$

But $\mathbf{dx}' = 0$ in K', and furthermore $\mathbf{dx}^2 = \mathbf{v}^2 dt^2$, and thus

$$cdt' = cdt\sqrt{1-\beta^2(t)},$$

where

$$\beta(t) = \frac{v(t)}{c}.$$

Then the elapsed time in the rocket between two events is

$$t_2' - t_1' = \int_{t_1}^{t_2} dt\sqrt{1-\beta^2(t)} < t_2 - t_1.$$

The **proper time** τ is the *elapsed time* in the frame in which the object is at rest. Thus

$$cd\tau = ds$$

where ds is the *interval* introduced earlier. In this case we have

$$d\tau = dt\sqrt{1-\beta^2(t)}\ . \tag{11.1.17}$$

Note that proper time can only be defined for *time-like* quantities.

11.1.10 Addition of Velocities

Suppose now that a projectile is fired with velocity $\mathbf{u}'$ from the rocket, relative to the rocket. Then the coordinates of the projectile in K' satisfy

$$\mathbf{u}' = \frac{d\mathbf{x}'}{dt'}.$$

while in K we have

$$\mathbf{u} = \frac{\mathbf{dx}}{dt}.$$

Using the Lorentz transformation with $v \to -v$, we have

$$\begin{aligned} x_\parallel &= \gamma_v[x_\parallel' + \beta ct'] \\ \Longrightarrow u_\parallel &\equiv \frac{dx_\parallel}{dt} = \gamma_v\left[\frac{dx_\parallel'}{dt'}\frac{dt'}{dt} + \beta c\frac{dt'}{dt}\right] \\ &= \gamma_v\left[\frac{dx_\parallel'}{dt'} + \beta c\right]\frac{dt'}{dt} = \gamma_v[u_\parallel' + v]\,\frac{dt'}{dt}\ , \end{aligned}$$

where we use $\|$ to denote the component along $\mathbf{v}$. We also have

$$\begin{aligned} ct &= \gamma_v[ct' + \beta x'_\|] \\ \Longrightarrow c &= \gamma_v\left[c\frac{dt'}{dt} + \beta u'_\| \frac{dt'}{dt}\right] = \gamma_v[c + \beta u'_\|]\,\frac{dt'}{dt}\ . \\ \Longrightarrow \frac{dt'}{dt} &= \frac{1}{\gamma_v[1 + \beta u'_\|/c]}\ . \end{aligned}$$

Combinding these two results, we find

$$u_\| = \frac{u'_\| + v}{1 + \beta u'_\|/c} = \frac{u'_\| + v}{1 + v u'_\|/c^2}\ . \tag{11.1.18}$$

Take $u'_\| = c$, then

$$u_\| = \frac{c + v}{1 + v/c} = c\ , \tag{11.1.19}$$

i.e., velocity of light is the same in both systems.
Similarly

$$u_\perp = \frac{dx_\perp}{dt} = \frac{dx'_\perp}{dt'} \cdot \frac{dt'}{dt},$$

yielding

$$u_\perp = \frac{u'_\perp}{\gamma(1 + \beta u'_\|/c)}. \tag{11.1.20}$$

In vector notation, this becomes

$$\begin{aligned} u_\| &= \frac{u'_\| + v}{1 + \mathbf{v}\cdot\mathbf{u}'/c^2} \\ \mathbf{u}_\perp &= \frac{\mathbf{u}'_\perp}{\gamma(1 + \mathbf{v}\cdot\mathbf{u}'/c^2)} \end{aligned} \tag{11.1.21}$$

As expected, this reduces to the Galilean result $\mathbf{u} = \mathbf{u}' + \mathbf{v}$ for the case $u', v \ll c$.
Let us use the "hyperbolic" parametrization for $u'_\|$ and v, namely

$$u'_\| = c\tanh\zeta'\ ,\ v = c\tanh\zeta_v\ . \tag{11.1.22}$$

Then Eq. (11.1.18) gives

$$\begin{aligned} u_\| = c\,\frac{\tanh\zeta' + \tanh\zeta_v}{1 + \tanh\zeta'\tanh\zeta_v} &= c\,\frac{\sinh\zeta'\cosh\zeta_v + \sinh\zeta_v\cosh\zeta'}{\cosh\zeta'\cosh\zeta_v + \sinh\zeta'\sinh\zeta_v} \\ &= c\,\frac{\sinh(\zeta' + \zeta_v)}{\cosh(\zeta' + \zeta_v)} = c\,\tanh(\zeta' + \zeta_v)\ , \end{aligned} \tag{11.1.23}$$

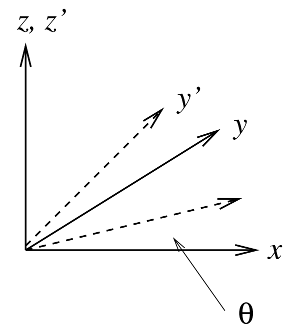
i.e., writing $u_\| = c\tanh\zeta$ we get $\zeta = \zeta' + \zeta_v$: it is the ζ-parameters (**rapidities**) which *simply add* when we *relativistically add* two parallel velocities.

11.2 Special Relativity and Four Vectors

We can formulate this picture in a much more convenient fashion through the introduction of *four vectors*. To see how these work, let us return briefly to Galilean transformations, and rotations in Euclidean space.

11.2.1 Vectors, Tensors and Rotations in R^3

Consider two co-ordinate systems P, P' whose origins coincide, but which are related by rotation through an angle θ.



The coordinates of a point in the two systems are related through

$$x'^i = R^i_j x^j, \tag{11.2.1}$$

where R is a rotation matrix. Note that we have put the indices **upstairs** on the vectors - we will return to this later. For the specific case of a rotation through θ about the z axis, the rotation matrix is

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Quantities that transform as

$$A'^i = R^i_j A^j = \frac{\partial x'^i}{\partial x^j} A^j \tag{11.2.2}$$

are called **vectors**.
A simple example of a vector is $d\mathbf{x}$, which transforms as

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j = R^i_j dx^j \ .$$

Scalars

A scalar is a quantity which transforms as $f' = f$.

Co-vectors or Forms

Let us now consider how the **gradient** of a function transforms:

$$\nabla'_i f = \frac{\partial f}{\partial x'^i} = \frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial f}{\partial x^j}.$$

This is an example of the transformation property

$$B'_i = \frac{\partial x^j}{\partial x'^i} B_j, \tag{11.2.3}$$

which is *different* from that of Eq. (11.2.2). Quantities that transform in this way are known as **covectors** or **forms**, and we put their indices downstairs.
Summarising, we have

$$\left.\begin{array}{rcl} \text{Vector:} \quad A'^i & = & \frac{\partial x'^i}{\partial x^j} A^j \\ \text{Scalar:} \quad f' & = & f \\ \text{Covector:} \quad B'_i & = & \frac{\partial x^j}{\partial x'^i} B_j \end{array}\right\} \tag{11.2.4}$$

Finally, we have that a **tensor** is an object that transforms as a *vector* on each *upstairs* index, and a *covector* on each *downstairs* index.

$$C'^{i'j'\dots}_{k'l'\dots} = \frac{\partial x'^{i'}}{\partial x^i}\frac{\partial x'^{j'}}{\partial x^j}\dots\frac{\partial x^k}{\partial x'^{k'}}\frac{\partial x^l}{\partial x'^{l'}}\dots C^{ij\dots}_{kl\dots}$$

Metric Tensor

The **length** of a vector is a bilinear, and independent of the choice of frame.
Define the **inner product** of two vectors by

$$X \cdot Y = g_{ij} X^i Y^j.$$

We call g_{ij} the $\boxed{\textbf{metric tensor}}$.
In Cartesian coordinates (x, y, z), we have

$$(dl)^2 = (dx)^2 + (dy)^2 + (dz)^2 \ , \tag{11.2.5}$$

hence,

$$g_{ij} = \delta_{ij}.$$

In spherical coordinates (r, θ, φ), we have

$$(dl)^2 = (dr)^2 + r^2(d\theta)^2 + r^2 \sin^2\theta (d\varphi)^2 \ , \tag{11.2.6}$$

hence

$$g_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta).$$

We can use the metric tensor to *raise* or *lower* indices:

$$\begin{aligned} X_i &= g_{ij} X^j \\ X \cdot Y &= X^i Y_i = X_i Y^i. \end{aligned}$$

We only have the luxury of indentifying *vectors* with *covectors* in Cartesian coordinates in Euclidean space, where the components of the two are numerically equal.
For instance, in spherical coordinates, taking $dx^i = \{dr, d\theta, d\varphi\}$ as a vector, we have $dx_i = \{dr, r^2 d\theta, r^2 \sin^2\theta d\varphi\}$ as the corresponding co-vector.

11.2.2 Minkowski Space-Time

We will now apply the above ideas to Lorentz transformations of four-dimensional space-time. We will introduce "*ct*" as the coordinate x_0, and write a **contravariant** four vector as

$$x^\mu \equiv (ct, x, y, z) = (x^0, x^1, x^2, x^3) \tag{11.2.7}$$

The "length" of the vector is the **interval** left invariant under Lorentz transformations. More generally, we define the inner product of two vectors by

$$x \cdot y = g_{\mu\nu} x^\mu y^\nu, \tag{11.2.8}$$

and we immediately see that

$$\boxed{g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \qquad (11.2.9)}$$

- Note that it is conventional to use *Greek Letters* for the components of a four-vector. Four vectors are not underlined or printed in bold.

- In some areas of physics, time is introduced as the *fourth* component of the vector. Furthermore, the metric can be defined such that the spatial components are positive, and the temporal component negative. The convention we will be using is probably the most widely used, and essentially universal amongst particle physicists.

- The summation convention is as follows:
  An index can appear no more than twice. Any index appearing twice must have one upper index and one lower index, and that index is summed over.

The $\boxed{\textbf{covariant four vector}}$ or **form** can be obtained as before by using the raising and lowering properties of the metric tensor

$$x_\mu = g_{\mu\nu} x^\nu .$$

In our example we have that $x_\mu = (ct, -x, -y, -z)$ – *the components of a co-vector are numerically different to those of the vector.*

11.2.3 Lorentz Transformations and Four Vectors

Let us return to our two frames K and K'. The relation between vectors (in 4-dimensional case of special relativity, they are called *contravariant vectors*) in the two frames is given by

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu = L^\mu{}_\nu x^\nu \tag{11.2.10}$$

Let us assume a similar transformation law for the 4-dimensional analogs of covectors (called *covariant vectors*)

$$x'_\mu = L_\mu{}^\nu x_\nu . \tag{11.2.11}$$

We require that $x^\mu x_\mu$ is invariant under the Lorentz transformation, i.e.

$$x^\mu x_\mu = x'^\mu x'_\mu = L^\mu{}_\nu L_\mu{}^\sigma x^\nu x_\sigma ,$$

and since this is true for all vectors, we have

$$\boxed{L^\mu{}_\nu L_\mu{}^\sigma = \delta_\nu^\sigma \ , \qquad (11.2.12)}$$

where

$$\delta_\nu{}^\sigma = \begin{cases} 1 & \text{if} \quad \nu = \sigma \\ 0 & \text{if} \quad \nu \neq \sigma \end{cases} \quad . \tag{11.2.13}$$

To find $L_\mu{}^\sigma$, we note that, according to (11.2.10), we have

$$L^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \ . \tag{11.2.14}$$

Now, using the identity

$$\frac{\partial x^\sigma}{\partial x^\nu} = \delta_\nu^\sigma \ . \tag{11.2.15}$$

written through the chain rule as

$$\delta_\nu^\sigma = \frac{\partial x^\sigma}{\partial x^\nu} = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\nu} = \underbrace{\frac{\partial x'^\mu}{\partial x^\nu}}_{L^\mu{}_\nu}\frac{\partial x^\sigma}{\partial x'^\mu} \ , \tag{11.2.16}$$

and comparing with Eq. (11.2.12), $L^{\mu}{}_{\nu} L_{\mu}{}^{\sigma} = \delta^{\sigma}_{\nu}$, we conclude that

$$L_{\mu}{}^{\sigma} = \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \;, \tag{11.2.17}$$

which corresponds to the characteristic transformation property of a form $\partial/\partial x^{\mu}$:

$$\frac{\partial f}{\partial x'^{\mu}} = \frac{\partial f}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x'^{\mu}} = \underbrace{\frac{\partial x^{\nu}}{\partial x'^{\mu}}}_{L_{\mu}{}^{\nu}} \frac{\partial f}{\partial x^{\nu}} \;.$$

Thus the various quantities we will encounter in the remainder of this course are

- **Contravariant Vectors**:
$$A'^{\mu} = L^{\mu}{}_{\nu} A^{\nu} \tag{11.2.18}$$

- **Covariant Vectors**:
$$B'_{\mu} = L_{\mu}{}^{\nu} B_{\nu} \tag{11.2.19}$$

- **Tensors**:
$$C'^{\mu'\nu'\ldots}{}_{\rho'\sigma'\ldots} = L^{\mu'}{}_{\mu} L^{\nu'}{}_{\nu} \ldots L_{\rho'}{}^{\rho} L_{\sigma'}{}^{\sigma} \ldots C^{\mu\nu\ldots}{}_{\rho\sigma\ldots} \tag{11.2.20}$$

- **Scalars**:
$$A \cdot B = A_{\mu} B^{\mu} = g_{\mu\nu} A^{\mu} B^{\nu}$$

As an excercize, let us demonstrate that $g_{\mu\sigma} A^{\sigma}$ is indeed a covector, i.e. transforms according to Eq. (11.2.19). We need to show that

$$\big(g_{\mu\sigma} A^{\sigma}\big)' \;=\; L_{\mu}{}^{\nu} \big(g_{\nu\lambda} A^{\lambda}\big)$$

The l.h.s. is

$$g_{\mu\sigma} (A^{\sigma})' \;=\; g_{\mu\sigma} L^{\sigma}{}_{\lambda} A^{\lambda}$$

and therefore we must prove that

$$g_{\mu\sigma} L^{\sigma}{}_{\lambda} \;=\; L_{\mu}{}^{\nu} g_{\nu\lambda} \;.$$

Multiplying this relation by $L_{\rho}{}^{\lambda}$ and using $L^{\sigma}{}_{\lambda} L_{\rho}{}^{\lambda} = \delta^{\sigma}_{\rho}$ converts it into

$$g_{\mu\rho} = g_{\nu\lambda} L_{\mu}{}^{\nu} L_{\rho}{}^{\lambda}$$

The last equation may be shown to follow from

$$\frac{\partial^2}{\partial x^{\mu} \partial x^{\rho}} g_{\nu\lambda} x^{\nu} x^{\lambda} \;=\; \frac{\partial^2}{\partial x^{\mu} \partial x^{\rho}} g_{\nu\lambda} x'^{\nu} x'^{\lambda}$$

and the definition $L_\mu^{\ \nu} \equiv \partial x'^\nu / \partial x^\mu$.
Indeed, we have, first, $g_{\nu\lambda} x^\nu x^\lambda = x^2$, $g_{\nu\lambda} x'^\nu x'^\lambda = x'^2$, and $x'^2 = x^2$. Then

$$\frac{\partial^2}{\partial x^\mu \partial x^\rho} g_{\nu\lambda} x^\nu x^\lambda = g_{\nu\lambda}[\delta^\nu_\mu \delta^\lambda_\rho + \delta^\nu_\rho \delta^\lambda_\mu] = 2 g_{\mu\rho} \ ,$$

and

$$\frac{\partial^2}{\partial x^\mu \partial x^\rho} g_{\nu\lambda} x'^\nu x'^\lambda = g_{\nu\lambda} \left[\underbrace{\frac{\partial x'^\nu}{\partial x^\mu}}_{L_\mu^{\ \nu}} \underbrace{\frac{\partial x'^\lambda}{\partial x^\rho}}_{L_\rho^{\ \lambda}} + \{\mu \leftrightarrow \rho\} \right] = g_{\nu\lambda}[L_\mu^{\ \nu} L_\rho^{\ \lambda} + L_\rho^{\ \nu} L_\mu^{\ \lambda}] = 2 g_{\nu\lambda} L_\mu^{\ \nu} L_\rho^{\ \lambda} \ .$$

On the last step we used the fact that $g_{\nu\lambda}$ is a symmetric tensor.
The metric tensor with upper indices $g^{\mu\nu}$ defines the inner product

$$x \cdot y = g^{\mu\nu} x_\mu y_\nu \tag{11.2.21}$$

in terms of covariant vectors. This product is invariant under Lorentz transformations if $g^{\mu\nu} y_\nu$ transforms as a contravariant vector y^μ. Using $y_\nu = g_{\nu\sigma} y^\sigma$, we conclude that

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma \ ,$$

i.e., the matrices $g^{\mu\nu}$ and $g_{\mu\nu}$ are inverse to each other.

11.2.4 Derivatives

As we have noted earlier, these transform as *covectors*

$$\begin{aligned} \partial_\alpha &= \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \nabla \right) \\ \partial^\alpha &= \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\nabla \right). \end{aligned} \tag{11.2.22}$$

Suppose now that we have a four vector A^μ. Then

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \nabla \cdot \mathbf{A}. \tag{11.2.23}$$

The 4D generalization of Laplacian (d'Alembertian) is defined by

$$\Box = \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x^{0 2}} - \nabla^2. \tag{11.2.24}$$

11.3 Relativistic Dynamics

11.3.1 Four Velocity

If we define the velocity in a usual way as $v^i = dx^i/dt$, and use that $t = x^0/c$, we immediately see that v^i cannot transform as a vector under Lorentz transformations. A formal reason is that such a derivative is a $0i$ component of a 4-tensor which does not transform as an i component of a 4-vector.

Indeed, let us assume that the object

$$\mathcal{V}^\mu \equiv \frac{dx^\mu}{dt} = \frac{d}{dt}\{x_0, \mathbf{x}\} = \frac{d}{dt}\{ct, \mathbf{x}\} = \{c, \mathbf{v}\}$$

transforms as a 4-vector. As usual, we consider two frames, the original K, and another, K', moving with velocity $\mathbf{V}$ with respect to K. Taking $\mathbf{V}$ along the x^3 axis and separating components parallel and transverse to $\mathbf{V}$, we write

$$\begin{aligned}\mathcal{V}^\mu =&\{c, \mathbf{0}_\perp, \mathcal{V}^3\}\\ \mathcal{V}'^\mu \equiv&\frac{dx'^\mu}{dt'} = \{c, \mathbf{0}_\perp, \mathcal{V}'^3\}\ .\end{aligned} \tag{11.3.1}$$

If $\mathcal{V}^\mu$ is a 4-vector, then, according to the Lorentz transformation, we should have

$$\mathcal{V}'^3 =\gamma_V\left(\mathcal{V}^3 - \frac{V}{c}\mathcal{V}^0\right) = \gamma_V\left(\mathcal{V}^3 - V\right)\ , \tag{11.3.2}$$

where $\gamma_V = 1/\sqrt{1 - V^2/c^2}$. This gives

$$\mathcal{V}'^3 = \left(\mathcal{V}^3 - V\right)/\sqrt{1 - V^2/c^2}$$

instead of the correct result that the velocity in the K' frame should be given by the formula

$$v'_\parallel = \frac{v_\parallel - V}{1 - v_\parallel V/c^2} \tag{11.3.3}$$

that follows from Eq. (11.1.18) for the relativistic velocity addition (in which we should take into account that K frame moves with respect to K' frame with the velocity $-\mathbf{V}$.)

So, the question is whether it is possible to find a definition of a velocity that does indeed transform covariantly under Lorentz transformations, yet reduces to a Galilean transformation for $v \ll c$?

In order to construct a **four velocity**, we need to take the derivative of the 4-vector x^μ with respect to some time that, unlike dt or dt', is the same in all frames, i.e. is a *Lorentz Scalar*. Such a scalar is provided by the **Proper Time** $d\tau$, or time measured in the frame that moves together with the particle, i.e. that has velocity $\mathbf{v}$ in the K frame. It is defined by

$$c^2 d\tau^2 = ds^2,$$

where ds is the Lorentz-invariant *interval*. The proper time is clearly a scalar, and therefore a natural definition of the **four velocity** is

$$\boxed{v^\alpha = \frac{dx^\alpha}{d\tau} \qquad (11.3.4)}$$

Recalling that the proper time is related to the K frame time by

$$d\tau = dt\,\sqrt{1-\beta^2(t)}$$

we have

$$v^\alpha = \frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt}(ct, \mathbf{x}) = \gamma(c, \mathbf{v}),$$

or

$$v^\alpha = (\gamma c, \gamma \mathbf{v})\ . \tag{11.3.5}$$

The spatial components of v^μ clearly reduce to our familiar definition of velocity in the non-relativistic (NR) limit. Note that $v^\alpha v_\alpha = c^2$.
Let us take now the component of $\mathbf{v}$ parallel to the relative velocity $\mathbf{V}$ and check that applying the Lorentz transformation to v^μ, namely

$$\begin{aligned} v'^3 =& \gamma_V \left(v^3 - \frac{V}{c} v^0\right) \\ v'^0 =& \gamma_V \left(v^0 - \frac{V}{c} v^3\right) \end{aligned} \tag{11.3.6}$$

leads to the correct relativistic velocity addition formula (11.3.3).
Indeed, substituting

$$v^0 = \gamma_v c \quad , \quad v^3 = \gamma_v v_\parallel$$

(where $\gamma_v = 1/\sqrt{1-v^2/c^2}$) and

$$v'^0 = \gamma_{v'} c \quad , \quad v'^3 = \gamma_{v'} v'_\parallel$$

(where $\gamma_{v'} = 1/\sqrt{1-v'^2/c^2}$), we rewrite (11.3.6) as

$$\begin{aligned} v'_\parallel \gamma_{v'} &= \gamma_V \gamma_v \left(v_\parallel - V\right) \\ c\gamma_{v'} &= \gamma_V \gamma_v \left(c - \frac{V}{c} v_\parallel\right)\ . \end{aligned} \tag{11.3.7}$$

It is easy to see that dividing the first of these equations by the second one gives

$$\frac{v'_\parallel}{c} = \frac{v_\parallel - V}{c - \frac{V}{c}v_\parallel} \quad \text{or} \quad v'_\parallel = \frac{v_\parallel - V}{1 - \frac{V}{c^2}v_\parallel}\ , \tag{11.3.8}$$

i.e., Eq. (11.3.3).

11.3.2 Four Momentum

The definition of a Lorentz-covariant 4-momentum is now straightforward:

$$p^\mu = mv^\mu = (m\gamma c, m\gamma\mathbf{v}), \tag{11.3.9}$$

where m is a **Lorentz scalar** that we will call the **rest mass**.
The spatial components of p^μ clearly reduce to our usual definition of momentum. To interpret the temporal component, we will look at its NR limit:

$$p^0 = m\gamma c = mc\left\{1 - v^2/c^2\right\}^{-1/2} = \frac{1}{c}\left\{mc^2 + \frac{1}{2}mv^2 + \mathcal{O}(v^4/c^2)\right\}.$$

The second term in braces is clearly the kinetic energy. The first term we identify as the **rest energy**, and write

$$p^0 = E/c$$

where E is the **energy**. Thus the four momentum contains both the energy and the three momentum.
The "length" of p^μ is a Lorentz scalar

$$\begin{aligned} p^\mu p_\mu &= m^2\gamma^2c^2 - m^2\gamma^2v^2 = m^2\gamma^2c^2\left[1 - v^2/c^2\right] \\ &= m^2\gamma^2c^2\gamma^{-2} = m^2c^2. \end{aligned}$$

Thus we have

$$\boxed{p^\mu p_\mu = p^2 = m^2c^2 \qquad (11.3.10)}$$

confirming that the rest mass is a (frame-independent) scalar.
Finally, if we now go back and write Eq. (11.3.10) in terms of our old-fashioned three vectors we have

$$\begin{aligned} \frac{1}{c^2}E^2 - \mathbf{p}^2 &= m^2c^2 \\ \Longrightarrow E^2 &= m^2c^4 + c^2\mathbf{p}^2. \end{aligned} \tag{11.3.11}$$

For a particle at rest, we have perhaps the most famous equation in physics

$$E = mc^2 \ .$$

The use of four-vectors is **essential** to solve problems in special (and general...) relativity. Whilst simple kinematical problems can be solved using three vectors, it is very clumsy indeed.

11.3.3 Energy-momentum conservation in application to $1 \to 2$ decay process

Consider a particle of mass M that decays at rest into two particles of masses m_1 and m_2. Energy momentum conservation requires that in any frame

$$P = p_1 + p_2 \ , \tag{11.3.12}$$

where P^μ is 4-momentum of the initial particle, $P^2 = M^2$, and $p_{1,2}$ are 4-momenta of final particles, $p_{1,2}^2 = m_{1,2}^2$. In the rest frame of the decaying particle we have

$$P = (M, \mathbf{0}) \ , \ p_1 = (E_1, \mathbf{p}_1) \ , \ p_2 = (E_2, \mathbf{p}_2) \ .$$

Hence,

$$E_1 + E_2 = M \qquad , \qquad \mathbf{p}_1 = -\mathbf{p}_2 \equiv \mathbf{p} \ . \tag{11.3.13}$$

Let us find first the energies of the final particles. Writing $p_2 = P - p_1$, we have

$$p_2^2 = P^2 - 2(Pp_1) + p_1^2 \ , \tag{11.3.14}$$

or

$$m_2^2 = M^2 - 2(ME_1 - \mathbf{0} \cdot \mathbf{p_1}) + m_1^2 \Rightarrow m_2^2 = M^2 - 2ME_1 + m_1^2 \ , \tag{11.3.15}$$

which gives

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} = \frac{1}{2}M + \frac{m_1^2 - m_2^2}{2M} \ . \tag{11.3.16}$$

In a similar way,

$$E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M} = \frac{1}{2}M - \frac{m_1^2 - m_2^2}{2M} \ . \tag{11.3.17}$$

Here $E_{1,2}$ are relativistic energies that include the rest mass term. The kinetic energy of the first final particle is given by

$$\begin{aligned}
E_1^{\text{kin}} =& \frac{M^2 + m_1^2 - m_2^2}{2M} - m_1 = \frac{M^2 - 2Mm_1 + m_1^2 - m_2^2}{2M} = \frac{(M - m_1)^2 - m_2^2}{2M} \\
=& \frac{(M - m_1 - m_2)(M - m_1 + m_2)}{2M} = \frac{\Delta M}{2}\left[1 - \frac{m_1 - m_2}{M}\right] \\
=& \Delta M \left[1 - \frac{\Delta M}{2M} - \frac{m_1}{M}\right] ,
\end{aligned} \tag{11.3.18}$$

where ΔM is the energy release. Similarly,

$$E_2^{\text{kin}} = \frac{\Delta M}{2}\left[1 + \frac{m_1 - m_2}{M}\right] = \Delta M \left[1 - \frac{\Delta M}{2M} - \frac{m_2}{M}\right] . \tag{11.3.19}$$

The magnitude of final particles' 3-momentum $|\mathbf{p}_1| = |\mathbf{p}_2| \equiv |\mathbf{p}|$ may be calculated from

$$\begin{aligned}
|\mathbf{p}|^2 =& E_1^2 - m_1^2 = \left(\frac{M^2 + m_1^2 - m_2^2}{2M}\right)^2 - m_1^2 \\
=& \frac{M^4 + m_1^4 + m_2^4 - 2m_1^2 m_2^2 - 2M^2 m_1^2 - 2M^2 m_2^2}{4M^2} \\
\equiv& \frac{\lambda(M^2, m_1^2, m_2^2)}{4M^2} ,
\end{aligned} \tag{11.3.20}$$

where

$$\lambda(a, b, c) \equiv (a + b - c)^2 - 4ab$$

is a symmetric function of all its three arguments. Thus, $|\mathbf{p}| = \sqrt{\lambda(M^2, m_1^2, m_2^2)}/2M$.
If the decay occurs in flight, then we can use

$$P^2 = p_1^2 + p_2^2 + 2(p_1 p_2) \tag{11.3.21}$$

or

$$M^2 = m_1^2 + m_2^2 + 2E_1E_2 - 2|\mathbf{p}_1||\mathbf{p}_2| \cos\theta , \tag{11.3.22}$$

where and θ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$ (see Problem 11.20 in Jackson, assigned for home work).

11.3.4 Energy-momentum conservation in application to $2 \to 2$ scattering process

Consider a process in which two initial particles with 4-momenta p_1, p_2 and masses m_1,m_2 convert into two final particles with 4-momenta p_3, p_4 and masses m_3,m_4. Using the momenta involved in this process, one can form several Lorentz invariants. First, we have four invariants involving one of the momenta: $p_1^2 = m_1^2$, $p_2^2 = m_2^2$, $p_3^2 = m_3^2$, $p_4^2 = m_4^2$. Combining momenta in pairs (and using the conservation law $p_1 + p_2 = p_3 + p_4$), we can form three *Mandelstam* invariants

$$(p_1 + p_2)^2 \equiv s \equiv (p_3 + p_4)^2 \ , \tag{11.3.23}$$

$$(p_1 - p_3)^2 \equiv t \equiv (p_2 - p_4)^2 \ , \tag{11.3.24}$$

$$(p_1 - p_4)^2 \equiv u \equiv (p_2 - p_3)^2 \ . \tag{11.3.25}$$

In fact, these invariants are not independent. There exists a linear relation

$$s + t + u = \sum_{i=1}^{4} m_i^2 \tag{11.3.26}$$

between them. Indeed,

$$\begin{aligned} s + t + u &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \\ &= m_1^2 + m_2^2 + 2(p_1 p_2) \\ &+ m_1^2 + m_3^2 - 2(p_1 p_3) \\ &+ m_1^2 + m_4^2 - 2(p_1 p_4) \\ &= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 \\ &+ 2p_1 \cdot \underbrace{(p_2 - p_3 - p_4)}_{-p_1} \\ &= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 - 2m_1^2 \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 \ . \end{aligned} \tag{11.3.27}$$

There are two natural frames to study this process. In the *laboratory* frame, the first particle is a projectile, $p_1 = (E_L, \mathbf{p}_L)$ and the second one is a target, $p_2 = (m_2, \mathbf{0})$. In the *center of mass* frame, the total 3-momentum of colliding particles is zero, i.e., $p_1 = (E_1, \mathbf{p})$, $p_2 = (E_2, -\mathbf{p})$. Since $s = (p_1 + p_2)^2$ is Lorentz invariant, we may write it in both systems. In particular, in laboratory frame we have

$$s = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2(p_1 p_2) = m_1^2 + m_2^2 + 2m_2 E_L \ . \tag{11.3.28}$$

This formula can be also obtained from

$$s = (m_2 + E_L)^2 - \mathbf{p}_L^2 \ . \tag{11.3.29}$$

In the center of mass frame, we have

$$s = (E_1 + E_2)^2 \equiv W^2 \ , \tag{11.3.30}$$

where $W \equiv \sqrt{s}$ is the total c.m. energy. Thus,

$$W^2 = m_1^2 + m_2^2 + 2m_2 E_L \ . \tag{11.3.31}$$

To get relation between the values of 3-momenta in these two frames, consider the scalar product $(p_1 p_2)$. Then

$$(p_1 p_2) = m_2 E_L = \mathbf{p}^2 + E_1 E_2 = \mathbf{p}^2 + \sqrt{(m_1^2 + \mathbf{p}^2)(m_2^2 + \mathbf{p}^2)} \tag{11.3.32}$$

or

$$(m_2 E_L - \mathbf{p}^2)^2 = (m_1^2 + \mathbf{p}^2)(m_2^2 + \mathbf{p}^2) \ , \tag{11.3.33}$$

which gives

$$\mathbf{p}^2 (m_1^2 + m_2^2 + 2m_2 E_L) = m_2^2 (E_L^2 - m_1^2) \ , \tag{11.3.34}$$

or

$$\mathbf{p}^2 W^2 = m_2^2 \mathbf{p}_L^2 \ . \tag{11.3.35}$$

Thus, $|\mathbf{p}| = |\mathbf{p}_L| m_2 / W$, and since $\mathbf{p}$ has the same direction as $\mathbf{p}_L$, we obtain

$$\mathbf{p} = \mathbf{p}_L \frac{m_2}{W} \ . \tag{11.3.36}$$

The magnitude of $|\mathbf{p}|$, c.m. 3-momentum of colliding particles, may be easily found by observation that the two initial particles with masses m_1, m_2 combine into a "particle" with mass $W = \sqrt{s}$, which is at rest in the c.m. frame. Hence, using Eq. (11.3.20), we get

$$|\mathbf{p}| = \frac{\sqrt{\lambda(s, m_1^2, m_2^2)}}{2\sqrt{s}} = \frac{\sqrt{(s - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2}}{2\sqrt{s}} \ . \tag{11.3.37}$$

In the final state, we have two particles with masses m_3,m_4 which originated from a "particle" with mass $\sqrt{s}$. Hence, the final particles in c.m. frame have opposite 3-momenta $\mathbf{p}'$, $-\mathbf{p}'$ whose magnitude is given by

$$|\mathbf{p}'| = \frac{\sqrt{\lambda(s, m_3^2, m_4^2)}}{2\sqrt{s}} = \frac{\sqrt{(s - m_3^2 - m_4^2)^2 - 4m_3^2 m_4^2}}{2\sqrt{s}} \ . \tag{11.3.38}$$

In general, there is an angle θ between the directions of $\mathbf{p}$ and $\mathbf{p}'$ (scattering angle in c.m. frame).
In particular case of **elastic scattering of identical particles,** when $m_i = m$, all c.m. energies E_i in this case are given by $W/2 = \sqrt{s}/2$, and

$$|\mathbf{p}| = |\mathbf{p}'| = \frac{\sqrt{(s-2m^2)^2 - 4m^4}}{2\sqrt{s}} = \frac{\sqrt{s-4m^2}}{2} \ . \tag{11.3.39}$$

In the laboratory frame, we have

$$s = 2m(E_L + m) \tag{11.3.40}$$

and $|\mathbf{p}_L| = |\mathbf{p}|\sqrt{s}/m$ or

$$|\mathbf{p}_L| = \frac{\sqrt{s(s-4m^2)}}{2m} \ . \tag{11.3.41}$$

The invariants t and u in c.m. variables in this case may be written as

$$t = (p_1 - p_3)^2 = -(\mathbf{p}_1 - \mathbf{p}_3)^2 = -2\mathbf{p}^2(1-\cos\theta) = -4\mathbf{p}^2\sin^2(\theta/2) \tag{11.3.42}$$

and

$$u = (p_1 - p_4)^2 = -(\mathbf{p}_1 + \mathbf{p}_3)^2 = -2\mathbf{p}^2(1+\cos\theta) = -4\mathbf{p}^2\cos^2(\theta/2) \ . \tag{11.3.43}$$

11.4 Covariant Formulation of Maxwell's Equation

Before considering Maxwell's equations in totality, we will return to the charge conservation.

11.4.1 Continuity Equation and Four Current

Charge conservation is expressed through the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{11.4.1}$$

We can write this in a more manifestly covariant form as

$$\frac{1}{c}\frac{\partial}{\partial t}(\rho c) + \nabla \cdot \mathbf{J} = 0.$$

It is therefore tempting to try to introduce a four-current

$$J^\mu = (\rho c, \mathbf{J}) \tag{11.4.2}$$

in terms of which Eq. (11.4.1) can be formally written as

$$\partial_\mu J^\mu = 0.$$

However, it remains to be shown that the J^μ thus constructed does indeed transform as a four vector.

Consider J^μ defined through Eq. (11.4.2) under a transformation to a frame K' moving with velocity v along the x axis. Then, if J^μ is indeed a four vector we would have

$$\begin{aligned}
\rho' c &= \gamma \left[\rho c - \frac{v}{c} J_x\right] \\
J'_x &= \gamma \left[J_x - v\rho\right] \\
J'_y &= J_y \\
J'_z &= J_z.
\end{aligned}$$

In the non-relativistic limit

$$\left.\begin{aligned}
\mathbf{J}' &= \mathbf{J} - \rho \mathbf{v} \\
\rho' &= \rho
\end{aligned}\right\},$$

as expected.

Consider now the case $J_x = 0$. Then we have

$$\left.\begin{aligned}
J'_x &= -\gamma v \rho \\
\rho' &= \gamma \rho
\end{aligned}\right\}.$$

The second equation would appear to violate charge conservation. However, let us consider what happens to a volume element under this transformation. In the frame K, we have

$$dV = dx\,dy\,dz.$$

However

$$\begin{aligned} dx &= \gamma(dx' + v\,dt') \\ dt &= \gamma(dt' + \frac{v}{c^2}\,dx') \\ dy &= dy' \\ dz &= dz'. \end{aligned}$$

Thus for measurements made at the same time ($dt' = 0$)

$$dV = dx\,dy\,dz = \gamma dx' dy' dz' = \gamma dV',$$

and the total charge in dV' is

$$\rho' dV' = \rho' \gamma^{-1} dV = \gamma\rho\gamma^{-1} dV = \rho\, dV$$

Thus both the charge densities and volumes are not separately conserved under this Lorentz transformation, but the charge itself is.
There is much experimental evidence that $\rho' = \gamma\rho$, and we will **postulate** that J^μ in Eq. (11.4.2) is indeed a four vector, and that

$$\partial_\mu J^\mu = 0 \tag{11.4.3}$$

11.4.2 Units

At this point, *Jackson* changes from SI to Gaussian units – the aim being to avoid carrying superfluous factors of c.

Gaussian Units

Below we present Maxwell's equations and relation between fields in Gaussian units:

$$\nabla \cdot \mathbf{D} = 4\pi\rho \tag{11.4.4}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \tag{11.4.5}$$

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \tag{11.4.6}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{11.4.7}$$

$$\mathbf{D} = \epsilon\mathbf{E} = \mathbf{E} + 4\pi\mathbf{P} \tag{11.4.8}$$

$$\mathbf{H} = \mathbf{B}/\mu = \mathbf{B} - 4\pi\mathbf{M} \tag{11.4.9}$$

You will notice that in these units $\partial/\partial t$ has an associated factor of $1/c$, corresponding to our definition of a four vector. Also, ϵ and μ are the *relative* permittivity and permeability respectively.

11.4.3 Potentials as Four Vectors

We introduce vector and scalar potentials so as to satisfy the homogeneous Maxwell equations

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ \mathbf{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \end{aligned} \tag{11.4.10}$$

In a vacuum ($\epsilon = \mu = 1$), the inhomogeneous equations become:

$$\begin{aligned} \nabla^2\phi + \frac{1}{c}\frac{\partial \nabla\cdot\mathbf{A}}{\partial t} &= -4\pi\rho \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left[\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right] &= -\frac{4\pi}{c}\mathbf{J}. \end{aligned}$$

In the **Lorentz gauge**, we have

$$\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0,$$

and the dynamical equations become

$$\begin{aligned} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} &= -4\pi\rho \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\frac{4\pi}{c}\mathbf{J}. \end{aligned} \tag{11.4.11}$$

We now recognize the operator on the l.h.s. of these equations as the four-dimensional Laplacian introduced in Eq. (11.2.24), and the r.h.s. as the temporal and spatial components of the current J^μ of Eq. (11.4.2). We will therefore introduce a four-vector potential

$$\mathbf{A}^\mu = (\phi, \mathbf{A}), \tag{11.4.12}$$

so that both equations in (11.4.11) can be unified in the manifestly covariant form

$$\Box A^\mu = \frac{4\pi}{c}J^\mu\ , \tag{11.4.13}$$

with

$$\Box \equiv \partial^\alpha\partial_\alpha = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\ .$$

Furthermore, the Lorentz gauge condition is also manifestly covariant:

$$\partial^\mu A_\mu = 0. \tag{11.4.14}$$

11.4.4 Field-Strength Tensor

In order to formulate the full Maxwell's equations in covariant form, we need to return to the relation between the fields $(\mathbf{E}, \mathbf{B})$ and the potentials

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ \mathbf{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \ . \end{aligned}$$

We need to find a covariant relation between electric and magnetic fields, and the four vector A^μ, and indeed express the fields themselves in covariant form. Let us write out a couple of these components explicitly

$$\begin{aligned} \mathbf{B}_x &= \frac{\partial \mathbf{A}_z}{\partial y} - \frac{\partial \mathbf{A}_y}{\partial z} = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3} = \frac{\partial A^2}{\partial x_3} - \frac{\partial A^3}{\partial x_2} \\ \mathbf{E}_x &= -\frac{\partial \phi}{\partial x} - \frac{1}{c}\frac{\partial \mathbf{A}_x}{\partial t} = -\frac{\partial A^0}{\partial x^1} - \frac{\partial A^1}{\partial x^0} = \frac{\partial A^0}{\partial x_1} - \frac{\partial A^1}{\partial x_0} \end{aligned}$$

or

$$B^1 = \partial^3 A^2 - \partial^2 A^3 = -(\partial^2 A^3 - \partial^3 A^2) = -\epsilon^{123}(\partial^2 A^3 - \partial^3 A^2) \quad \rightarrow \quad -\epsilon^{123} F^{23}$$
$$E^1 = \partial^1 A^0 - \partial^0 A^1 \quad \rightarrow \quad F^{10} \ .$$

N.B.: We are using here a slightly confusing notation: $\mathbf{E}_i$ denotes the i^{th} component of a **three vector**, where we do not need to distinguish between covariant and contravariant vectors. The equivalent four-vector components are given by

$$\begin{aligned} E^i &= \mathbf{E}_i \\ E_i &= -\mathbf{E}_i . \end{aligned}$$

We can see that $(\mathbf{E}, \mathbf{B})$ are related to a second-rank tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, and there are six independent components of the two fields.
For a general second-rank tensor $T^{\mu\nu}$, we can write

$$T^{\mu\nu} = T^{\mu\nu}_{\text{sym}} + T^{\mu\nu}_{\text{anti-sym}} .$$

The symmetric part has ten components, but the anti-symmetric part has six independent components that we could associate with fields $\mathbf{E}$ and $\mathbf{B}$. Thus we introduce the anti-symmetric **Maxwell Field-Strength Tensor**

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad (11.4.15)}$$

Writing out the components of $F^{\mu\nu}$ explicitly, we have (μ numbers rows, from 0 to 3, and ν numbers columns)

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} , \tag{11.4.16}$$

or

$$F^{i0} = E^i \ ; \ F^{0i} = -E^i$$

($i = 1, 2, 3$) and

$$F^{ij} = -\epsilon^{ijk} B^k \ ,$$

($i, j, k = 1, 2, 3$) with ϵ^{ijk} being the 3-dimensional Levi-Civita tensor (note that, with chosen non-equal i, j, there is only one possibility for k in ϵ^{ijk}). We see that **E** and **B** are not components of four vectors, but rather of an anti-symmetric, second-rank tensor. Note that we can lower the indices in the usual way

$$F_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta},$$

so that the components corresponding to **E** change sign, $F_{i0} = -E^i$, whilst those corresponding to **B** are unaltered:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} . \tag{11.4.17}$$

Inverting the relation involving B^k gives

$$B^k = -\frac{1}{2} \epsilon^{ijk} F_{ij} \ .$$

Here, the summation over both i and j index is implied. Note that, with a chosen k, there are two possibilities for non-equal i, j in ϵ^{ijk}.
Finally, we will introduce the **dual** field-strength tensor. But as a precursor we will return to the Levi-Civita tensor.

Levi-Civita Tensor in 4 dimensions

This is the four-dimensional version of the ϵ_{ijk} encountered in 3-D Euclidean space. It is defined by

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } \mu, \nu, \rho, \sigma \text{ is an } \textit{even} \text{ perm of } 0, 1, 2, 3 \\ -1 & \text{if } \mu, \nu, \rho, \sigma \text{ is an } \textit{odd} \text{ perm of } 0, 1, 2, 3 \\ 0 & \text{if any two indices are equal} \end{cases} \tag{11.4.18}$$

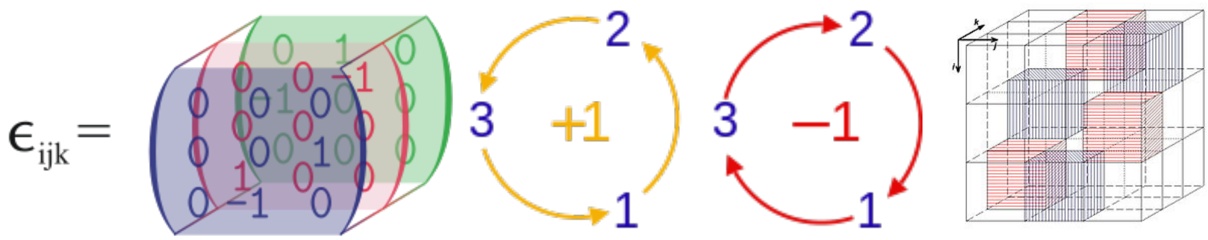


Figure 11.1: Visualizations of 3D Levi-Civita symbol

Lowering the indices in the usual way, we immediately see that

$$\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}.$$

Note a very useful relation

$$\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2\left(\delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma} - \delta_{\sigma}{}^{\mu}\delta_{\rho}{}^{\nu}\right). \tag{11.4.19}$$

If we take $\mu = 0$, then other components of $\epsilon^{\mu\nu\rho\sigma}$ are space-like, and

$$\epsilon^{0ijk} = \epsilon^{ijk} \ .$$

As an exercise, let us assume all indices μ, ν, ρ, σ in Eq. (11.4.19) correspond to space components m, n, r, s. Then one of the α, β indices corresponds to the time component, i.e. either $\alpha = 0$ or $\beta = 0$, and the remaining one corresponds to a space component. Then the left side is

$$\epsilon^{\alpha\beta mn}\epsilon_{\alpha\beta rs} = \epsilon^{0bmn}\epsilon_{0brs} + \epsilon^{a0mn}\epsilon_{a0rs} = -\epsilon^{bmn}\epsilon^{brs} - \epsilon^{amn}\epsilon^{ars} = -2\epsilon^{bmn}\epsilon^{brs} = -2\epsilon^{mnb}\epsilon^{brs} \ . \tag{11.4.20}$$

The right side:

$$-2\left(\delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma} - \delta_{\sigma}{}^{\mu}\delta_{\rho}{}^{\nu}\right) \Rightarrow -2\left(\delta^{mr}\delta^{ns} - \delta^{sm}\delta^{rn}\right). \tag{11.4.21}$$

Thus, Eq. (11.4.19) gives

$$\epsilon^{mnb}\epsilon^{brs} = \delta^{mr}\delta^{ns} - \delta^{sm}\delta^{rn} \ . \tag{11.4.22}$$

Multiplying by $A^n B^r C^s$ gives

$$\begin{aligned}\epsilon^{mnb}A^n \underbrace{\epsilon^{brs}B^r C^s}_{(\mathbf{B}\times\mathbf{C})^b} &= \epsilon^{mnb}A^n(\mathbf{B}\times\mathbf{C})^b = [\mathbf{A}\times(\mathbf{B}\times\mathbf{C}]^m \\ &= [\delta^{mr}\delta^{ns} - \delta^{sm}\delta^{rn}]A^n B^r C^s = B^m(\mathbf{A}\cdot\mathbf{C}) - C^m(\mathbf{A}\cdot\mathbf{B}) \ ,\end{aligned} \tag{11.4.23}$$

i.e. the $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$ formula.

11.4.5 Dual Field-Strength Tensor

The dual field-strength tensor is defined by

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{11.4.24}$$

Take $\mu = 0$, then

$$\tilde{F}^{0i} = \frac{1}{2}\epsilon^{0ijk}F_{jk} = \frac{1}{2}\epsilon^{ijk}F_{jk} = -B^i \ ,$$

or $B^i = \tilde{F}^{i0}$. Similarly, taking both μ and ν space-like, we have

$$\tilde{F}^{ij} = \frac{1}{2}\left[\epsilon^{ij0k}F_{0k} + \epsilon^{ijk0}F_{k0}\right] = \epsilon^{ijk}F_{0k} = \epsilon^{ijk}E^k \ .$$

Thus, the elements of $\tilde{F}^{\mu\nu}$ are related to those of $F^{\mu\nu}$ through the substitution

$$\begin{aligned} \mathbf{E} &\to \mathbf{B} \\ \mathbf{B} &\to -\mathbf{E}, \end{aligned}$$

so that

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}.$$

Thus transition from $F^{\mu\nu}$ to $\tilde{F}^{\mu\nu}$ reverses the roles of the electric and magnetic fields. Lowering the indices converts $\mathbf{B} \to -\mathbf{B}$, $\mathbf{E} \to \mathbf{E}$, and we have

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}.$$

11.4.6 Maxwell's Equations

Let us return to Maxwell's equation in a vacuum

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{11.4.25}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \tag{11.4.26}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{11.4.27}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{11.4.28}$$

These are all first-order differential equations expressed in terms of $\mathbf{E}$ and $\mathbf{B}$. Thus we might suspect that the covariant form of Maxwell's equations will contain terms of the form $\partial_\mu F_{\nu\rho}$. To convert equations written for the fields $\mathbf{E}, \mathbf{B}$ into equations for the tensor $F_{\nu\rho}$, we will use the relations $E^i = F^{i0} = -F^{0i}$, $F^{ij} = -\epsilon^{ijk}B^k$, $B^k = -\frac{1}{2}\epsilon^{ijk}F_{ij}$ and $B^i = \tilde{F}^{i0}$, $\tilde{F}^{ij} = \epsilon^{ijk}E^k$ found earlier.

Looking at Eq. (11.4.25), we see that it may be written as

$$\frac{\partial}{\partial x^i}E^i = 4\pi\frac{J^0}{c}.$$

Using $E^i = F^{i0}$, and noting that F^{00} vanishes, we can rewrite (11.4.25) as

$$\partial_\mu F^{\mu 0} = \frac{4\pi}{c}J^0. \tag{11.4.29}$$

Turning now to the second inhomogeneous equation, Eq. (11.4.27), we see that it may be written as

$$\epsilon^{ijk}\frac{\partial}{\partial x^j}B^k = \frac{4\pi}{c}J^i + \frac{1}{c}\frac{\partial}{\partial t}E^i\ .$$

Using $\epsilon^{ijk}B^k = -F^{ij} = F^{ji}$ and $E^i = -F^{0i}$ gives

$$\frac{\partial}{\partial x^j}F^{ji} = \frac{4\pi}{c}J^i - \frac{1}{c}\frac{\partial}{\partial t}F^{0i}\ .$$

Thus Eq. (11.4.27) can be written as

$$\begin{aligned} \frac{\partial}{\partial x^j}F^{ji} + \frac{\partial}{\partial x^0}F^{0i} &= \frac{4\pi}{c}J^i && (11.4.30)\\ \Longrightarrow \partial_\mu F^{\mu i} &= \frac{4\pi}{c}J^i. && (11.4.31)\end{aligned}$$

Thus we see that the two inhomogeneous Maxwell equations can be written in the unified form

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu\ . \qquad (11.4.32)}$$

Turning now to the homogeneous equations, we see that Eq. (11.4.28) can be written as

$$\begin{aligned} \frac{\partial}{\partial x^i}\tilde{F}^{i0} &= 0\\ \Longrightarrow \partial_\mu \tilde{F}^{\mu 0} &= 0. \end{aligned}$$

Eq. (11.4.26) takes the form

$$\epsilon^{ijk}\frac{\partial}{\partial x^j}E^k + \frac{1}{c}\frac{\partial}{\partial t}B^i = 0$$

Using $\epsilon^{ijk}E^k = \tilde{F}^{ij}$ and $B^i = \tilde{F}^{i0}$ we obtain

$$\begin{aligned}\frac{\partial}{\partial x^j}\tilde{F}^{ij} + \frac{\partial}{\partial x^0}\tilde{F}^{i0} &= 0 \\ \Longrightarrow \partial_\mu \tilde{F}^{\mu i} &= 0.\end{aligned}$$

Thus the two homogeneous Maxwell equations can be written in the unified form

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0 \ . \qquad (11.4.33)}$$

Eqns. (11.4.32) and (11.4.33) constitute the covariant formulation of Maxwell's equations.

Note that we can rewrite Eq. (11.4.33) as

$$\begin{aligned}\frac{1}{2}\partial_\mu \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} &= 0 \\ \Longrightarrow \epsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} &= 0,\end{aligned}$$

which we can express as

$$\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho} = 0. \qquad (11.4.34)$$

This is known as the **Jacobi Identity**.

11.4.7 Energy and Momentum Law

The Lorentz force law in Gaussian units is

$$\frac{d\mathbf{p}}{dt} = q\left\{\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right\}.$$

In order to write this in a covariant form, we introduce the proper time

$$d\tau = \gamma^{-1}dt,$$

and write

$$\frac{dp^i}{dt} = \frac{dp^i}{d\tau}\frac{d\tau}{dt} = \frac{1}{\gamma}\frac{dp^i}{d\tau}.$$

Thus the force law may be expressed as

$$\frac{dp^i}{d\tau} = \gamma q\left\{E^i + \frac{1}{c}\epsilon^{ijk}v^j B^k\right\}.$$

We now introduce the four-velocity $V^\mu = (\gamma c, \gamma \mathbf{v})$, yielding

$$\begin{aligned} \frac{dp^i}{d\tau} &= \frac{q}{c}\{V^0 F^{i0} + \epsilon^{ijk} V^j B^k\} \\ &= \frac{q}{c}\{V^0 F^{i0} - V^j F^{ij}\} \\ &= \frac{q}{c}\{V_0 F^{i0} + V_j F^{ij}\}\ . \end{aligned}$$

Thus the Lorentz force law becomes

$$\frac{dp^i}{d\tau} = \frac{q}{c}\, V_\mu F^{i\mu}. \tag{11.4.35}$$

The analogous equation for the energy is

$$\frac{d}{dt} E^{\mathrm{mech}} = q\mathbf{E}\cdot\mathbf{v}.$$

Thus, writing

$$\frac{d}{dt}E^{\mathrm{mech}} = \frac{d\tau}{dt}\,\frac{d}{d\tau}E^{\mathrm{mech}} = \frac{1}{\gamma}\,\frac{d}{d\tau}E^{\mathrm{mech}}$$

we have

$$\begin{aligned} \frac{dE^{\mathrm{mech}}}{d\tau} &= \gamma q F^{i0} v^i \\ &= q F^{0i} V_i, \end{aligned}$$

yielding

$$\frac{d}{d\tau}\left(\frac{E^{\mathrm{mech}}}{c}\right) = \frac{q}{c} V_\mu F^{0\mu}.$$

Identifying E^{mech}/c with the component p^0, we see that both this equation and the Lorentz law, Eq. (11.4.35), can be expressed as

$$\boxed{\frac{dp^\mu}{d\tau} = \frac{q}{c} V_\nu F^{\mu\nu} \qquad (11.4.36)}$$

and Newton's second law is in a manifestly covariant form.

11.4.8 Lorentz Invariants

There are two invariants we can construct from the field-strength tensor

1.
$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} \\ &= 2(\mathbf{B}^2 - \mathbf{E}^2) \ . \end{aligned}$$

   On the last step, we used explicit forms of $F^{\mu\nu}$ and $F_{\mu\nu}$:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad , \quad F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} . \tag{11.4.37}$$

   Thus
$$\mathbf{B}^2 - \mathbf{E}^2 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} \tag{11.4.38}$$
   is a **Lorentz Scalar**.

2. The second invariant is given by

$$\begin{aligned} F_{\mu\nu}\tilde{F}^{\mu\nu} =& F_{0i}\tilde{F}^{0i} + F_{i0}\tilde{F}^{i0} + F_{ij}\tilde{F}^{ij} = -2\mathbf{E}\cdot\mathbf{B} - \epsilon_{ijk}B^k\epsilon_{ijl}E^l \\ &= -2\mathbf{E}\cdot\mathbf{B} - 2\delta_{kl}B^kE^l = -4\mathbf{E}\cdot\mathbf{B} \ , \end{aligned} \tag{11.4.39}$$

   where we used $\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$. The result can be checked by using explicit form of $\tilde{F}^{\mu\nu}$:

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} .$$

   Thus
$$\mathbf{E}\cdot\mathbf{B} = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{11.4.40}$$
   is also a **Lorentz Scalar** (more precisely, a *pseudoscalar*).

These are the *only* Lorentz invariants built from electromagnetic fields.

11.5 Transformation Properties of EM Field

Since $F^{\mu\nu}$ is a second-rank tensor, we can immediately say it transforms according to

$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} F^{\alpha\beta} \frac{\partial x'^{\nu}}{\partial x^{\beta}},$$

which we can write as

$$F' = \Lambda F \Lambda^T, \tag{11.5.1}$$

where

$$\Lambda^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}.$$

Specifically, let us consider a boost from K to K' where K' has velocity v in x-direction w.r.t. K, and origins coincide at $t = t' = 0$. Then

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$. Using this expression in Eq. (11.5.1), we find

$$\begin{aligned} F' &= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma\beta E_1 & -\gamma E_1 & -E_2 & -E_3 \\ \gamma E_1 & -\gamma\beta E_1 & -B_3 & B_2 \\ \gamma(E_2 - \beta B_3) & \gamma(B_3 - \beta E_2) & 0 & -B_1 \\ \gamma(E_3 + \beta B_2) & -\gamma(B_2 + \beta E_3) & B_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -E_1 & -\gamma(E_2 - \beta B_3) & -\gamma(E_3 + \beta B_2) \\ E_1 & 0 & -\gamma(B_3 - \beta E_2) & \gamma(B_2 + \beta E_3) \\ \gamma(E_2 - \beta B_3) & \gamma(B_3 - \beta E_2) & 0 & -B_1 \\ \gamma(E_3 + \beta B_2) & -\gamma(B_2 + \beta E_3) & B_1 & 0 \end{pmatrix}. \end{aligned} \tag{11.5.2}$$

Writing out the individual vector components, we find

$$\left.\begin{array}{lllllll} E'_1 & = & E_1 & ; & B'_1 & = & B_1 \\ E'_2 & = & \gamma(E_2 - \beta B_3) & ; & B'_2 & = & \gamma(B_2 + \beta E_3) \\ E'_3 & = & \gamma(E_3 + \beta B_2) & ; & B'_3 & = & \gamma(B_3 - \beta E_2) \end{array}\right\} \tag{11.5.3}$$

Thus the **E** and **B** fields **mix** under a Lorentz transformation.
We can express this in (three) vector form as

$$\begin{aligned}\mathbf{E}' &= \gamma[\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}] - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E}) \\ \mathbf{B}' &= \gamma(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B}),\end{aligned} \tag{11.5.4}$$

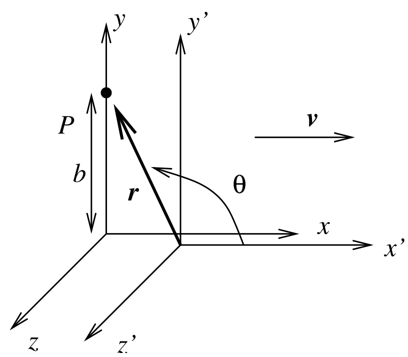
where $\beta = \mathbf{v}/c$.
In particular, take the component of $\mathbf{E}'$ parallel to $\mathbf{v}$. This gives

$$E_1' = \gamma E_1 - \gamma^2\beta^2/(\gamma+1)E_1 = \frac{\gamma^2 + \gamma - \gamma^2\beta^2}{\gamma+1}\,E_1 = \frac{1+\gamma}{\gamma+1}\,E_1 = E_1$$

since $\gamma^2 - \gamma^2\beta^2 = 1$.

11.5.1 Electric and magnetic fields of relativistically moving point charge.

Consider a charge q moving along a line at velocity (in K) $\mathbf{v} = v\mathbf{e_1}$. The charge is at rest in the frame K'.



At $t = t' = 0$, the origins of the two frames coincide. We have an observer P at impact parameter b (i.e. distance of closest approach) as shown above.
We will begin by looking at electric and magnetic fields at point P *in frame* K' at time t'.
P has coordinates

$$\begin{aligned}x' &= -vt' \\ y' &= b \\ z' &= 0.\end{aligned}$$

Thus, from Coulomb's law

$$\begin{array}{rclcrclcrcl} E_1' &=& -qvt'/r'^3 &;& E_2' &=& qb/r'^3 &;& E_3' &=& 0 \\ B_1' &=& 0 &;& B_2' &=& 0 &;& B_3' &=& 0. \end{array}$$

In order to express this in terms of coordinates in K, we note that $r'^2 = b^2 + v^2t'^2$. But we have

$$ct' = \gamma(ct - \beta x) = \gamma ct.$$

Thus

$$r'^2 = b^2 + v^2\gamma^2 t^2$$

and we have

$$\begin{aligned} E_1' &= -\frac{q\gamma vt}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \\ E_2' &= \frac{qb}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \\ E_3' &= 0. \end{aligned}$$

We now use our transformation laws Eq. (11.5.3) changing there $\beta \to -\beta$:

$$\left.\begin{array}{rclcrcl} E_1 &=& E_1' &;& B_1 &=& B_1' \\ E_2 &=& \gamma(E_2' + \beta B_3') &;& B_2 &=& \gamma(B_2' - \beta E_3') \\ E_3 &=& \gamma(E_3' - \beta B_2') &;& B_3 &=& \gamma(B_3' + \beta E_2') \end{array}\right\} \tag{11.5.5}$$

to write

$$\begin{aligned} E_1 &= E_1' = -\frac{q\gamma vt}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \\ E_2 &= \gamma E_2' = \frac{\gamma qb}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \\ E_3 &= \gamma E_3' = 0 \\ B_1 &= 0; B_2 = \gamma B_2' = 0 \\ B_3 &= \gamma\beta E_2' = \beta E_2 \end{aligned}$$

Thus in the laboratory frame we see a magnetic induction.
Note that in the limit $v \to c$, we have $\beta \to 1$ and the magnetic induction equals the transverse electric field. In the Galilean limit $v \to 0$,

$$\begin{aligned} B_3 &= \frac{v}{c}\frac{\gamma qb}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \longrightarrow \frac{vqb}{c(b^2 + v^2 t^2)^{3/2}} \\ \Longrightarrow \mathbf{B} &\sim \frac{q}{c}\frac{\mathbf{v} \times \mathbf{r}}{r^3} \end{aligned}$$

where we have used $vb = vr\sin\theta$, which we observe is just the *Biot-Savart Law*.
Finally, let us look at the field lines. We have that

$$\frac{E_2}{E_1} = -\frac{b}{vt},$$

so that the electric field is still a central field in the frame K. If we now look at the *magnitude* of the field, however, we find

$$|\mathbf{E}| = \frac{\gamma q}{(b^2 + v^2\gamma^2 t^2)^{3/2}}(b^2 + v^2 t^2)^{1/2}.$$

Setting $b = r\sin\theta, vt = r\cos\theta$, we have

$$|\mathbf{E}| = \frac{\gamma q r}{r^3(\sin^2\theta + \gamma^2\cos^2\theta)^{3/2}} = \frac{q}{r^2\gamma^2(\sin^2\theta/\gamma^2 + \cos^2\theta)^{3/2}} = \frac{q}{\gamma^2 r^2}(1 - \beta^2\sin^2\theta)^{-3/2}.$$

So the lines of force, whilst central, are no longer *isotropic*.

11.5.2 Plane Electromagnetic Radiation and Doppler Shift

Let us look at the propagation of a plane wave in vacuum. Our starting point is the Jacobi identity Eq. (11.4.34). Applying ∂^α we find

$$\partial^\alpha\partial_\alpha F_{\beta\gamma} + \partial_\beta\partial^\alpha F_{\gamma\alpha} + \partial_\gamma\partial^\alpha F_{\alpha\beta} = 0. \tag{11.5.6}$$

In the absence of sources,

$$\partial^\mu F_{\mu\nu} = \frac{4\pi}{c}J_\nu = 0.$$

Thus the last two terms on the r.h.s. of Eq. (11.5.6) vanish, and we have the plane e.m. waves satisfy

$$\boxed{\partial^\alpha\partial_\alpha F_{\mu\nu} \equiv \Box F_{\mu\nu} = 0\ , \qquad (11.5.7)}$$

with

$$\Box \equiv \partial^\alpha\partial_\alpha = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

being the differential operator of the wave equation. In complete analogy to the three-dimensional NR formulations, we note that this admits the solution

$$F_{\mu\nu} = f_{\mu\nu}e^{ik_\alpha x^\alpha} \tag{11.5.8}$$

where $k_\alpha k^\alpha = k^2 = 0$. Writing $k^\alpha = (\omega/c, \mathbf{k})$, we see that $k^2 = 0$ is just $\mathbf{k}^2 = \omega^2/c^2$ which is our usual relation between wave number and frequency.

We will now look at the transformation properties of the solution. We will let the solution in frame K be

$$F_{\mu\nu} = f_{\mu\nu} e^{ik\cdot x}$$

whilst that in K' be

$$F'_{\mu\nu} = f'_{\mu\nu} e^{ik'\cdot x'}.$$

The solutions in the two frames are related by

$$F'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}.$$

This can be satisfied $\forall x$ iff $k' \cdot x' = k \cdot x$ showing that k and k' are indeed four vectors. Because of this, we know that k_μ and k'_μ are related by

$$\begin{aligned} k'_\parallel &= \gamma[k_\parallel - \beta k_0] \\ k'_0 &= \gamma[k_0 - \beta k_\parallel] \\ \mathbf{k}'_\perp &= \mathbf{k}_\perp \end{aligned} \tag{11.5.9}$$

Introducing θ as the angle between $\mathbf{k}$ and $\mathbf{v}$, we can use the second equation of (11.5.9) to compute the Doppler shift:

$$\frac{\omega'}{c} = \gamma\left[\frac{\omega}{c} - \frac{v}{c}|\mathbf{k}|\cos\theta\right] = \gamma\left[\frac{\omega}{c} - \frac{v}{c}\frac{\omega}{c}\cos\theta\right].$$

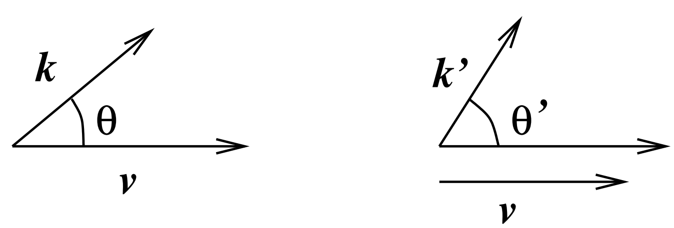
Thus we have the Doppler Shift formula

$$\boxed{\omega' = \gamma\omega(1 - \beta\cos\theta) \qquad (11.5.10)}$$

where $\beta = v/c$. This is modified from the usual Galilean formula through the factor of γ.

11.5.3 Aberration

This is the change in *direction* of a wave vector between the two frames.



We can calculate this from

$$\tan\theta' = \frac{|\mathbf{k}'_\perp|}{k'_\parallel} = \frac{|\mathbf{k}_\perp|}{k'_\parallel}.$$

By our Lorentz transformation formula

$$\begin{aligned} k'_\parallel &= \gamma\left[k_\parallel - \beta\frac{\omega}{c}\right] = \gamma\left[\frac{\omega}{c}\cos\theta - \beta\frac{\omega}{c}\right] \\ &= \gamma\frac{\omega}{c}(\cos\theta - \beta) \end{aligned}$$

Also we have

$$\mathbf{k}_\perp^2 = k_0^2 - k_\parallel^2 = \left(\frac{\omega}{c}\right)^2(1-\cos^2\theta) = \left(\frac{\omega}{c}\sin\theta\right)^2,$$

and thus

$$|\mathbf{k}_\perp| = \frac{\omega}{c}\sin\theta.$$

Thus, taking the ratio, we find

$$\boxed{\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)} \qquad (11.5.11)}$$

11.6 Motion of Particles in Electromagnetic Fields

In this section we will discuss several examples of motion in electric and/or magnetic fields.

11.6.1 Motion in a constant homogeneous electric field

Consider a constant uniform electric field $\mathbf{E}$. The equation of motion

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E} \tag{11.6.1}$$

will be simplified if we take $\mathbf{E}$ in x-direction. Then

$$\dot{p}_x = eE \qquad , \qquad \dot{p}_y = 0 \ , \tag{11.6.2}$$

which, assuming $p_x = 0, p_y = p_0$ for $t = 0$, gives

$$p_x = eEt, \qquad p_y = p_0 \ . \tag{11.6.3}$$

The energy of the particle is then

$$\mathcal{E} = c\sqrt{m^2c^2 + p^2} = \sqrt{m^2c^4 + c^2p_0^2 + (ceEt)^2} = \sqrt{\mathcal{E}_0^2 + (ceEt)^2} \ , \tag{11.6.4}$$

where $\mathcal{E}$ is energy at $t = 0$. For velocity of the particle, we have

$$\mathbf{v} = \mathbf{p}\frac{c^2}{\mathcal{E}} \tag{11.6.5}$$

or

$$\frac{dx}{dt} = \frac{p_x c^2}{\mathcal{E}} = \frac{c^2eEt}{\sqrt{\mathcal{E}_0^2 + (ceEt)^2}} \ . \tag{11.6.6}$$

Integrating this equation over time we obtain

$$x - x_0 = c^2eE\int_0^t \frac{tdt}{\sqrt{\mathcal{E}_0^2 + (ceEt)^2}} \ . \tag{11.6.7}$$

Introducing variable $\xi = (ceEt)^2$, we have

$$x - x_0 = c^2eE\frac{1}{2(ceE)^2}\int_0^{(ceEt)^2} \frac{d\xi}{\sqrt{\mathcal{E}_0^2 + \xi}} = \frac{1}{eE}\left(\sqrt{\mathcal{E}_0^2 + (ceEt)^2} - \mathcal{E}_0\right) \ . \tag{11.6.8}$$

For motion in y-direction, we have

$$\frac{dy}{dt} = \frac{p_y c^2}{\mathcal{E}} = \frac{p_0c^2}{\sqrt{\mathcal{E}_0^2 + (ceEt)^2}} \ , \tag{11.6.9}$$

which gives

$$y - y_0 = p_0 c^2 \int_0^t \frac{dt}{\sqrt{\mathcal{E}_0^2 + (ceEt)^2}} = \frac{p_0 c^2}{ceE} \int_0^{ceEt} \frac{d\eta}{\sqrt{\mathcal{E}_0^2 + \eta^2}} = \frac{p_0 c}{eE} \sinh^{-1}\left(\frac{ceEt}{\mathcal{E}_0}\right) . \tag{11.6.10}$$

Inverting this relation we arrive at

$$ceEt = \mathcal{E}_0 \sinh\left(\frac{eE(y - y_0)}{p_0 c}\right) . \tag{11.6.11}$$

As a result, we have equation for the trajectory

$$x - x_0 = \frac{\mathcal{E}_0}{eE}\left[\cosh\left(\frac{eE(y - y_0)}{p_0 c}\right) - 1\right] . \tag{11.6.12}$$

In the nonrelativistic limit $c \to \infty$ it may be written as

$$\begin{aligned} x - x_0 =& \frac{\mathcal{E}_0}{eE}\left[\frac{1}{2}\left(\frac{eE(y - y_0)}{p_0 c}\right)^2 + \mathcal{O}(1/c^4)\right] \\ =& \frac{eE(y - y_0)^2}{2mv_0^2} + \mathcal{O}(1/c^2) , \end{aligned} \tag{11.6.13}$$

i.e. the motion curve is represented by a parabola.

11.6.2 Motion in a uniform static magnetic field

Consider now a constant uniform magnetic field $\mathbf{B}$. When both electric and magnetic fields are present, the equations of motion are

$$\frac{d\mathbf{p}}{dt} = e\left[\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right] \tag{11.6.14}$$

$$\frac{d\mathcal{E}}{dt} = e\,\mathbf{v} \cdot \mathbf{E} . \tag{11.6.15}$$

In a purely magnetic field,

$$\frac{d\mathbf{p}}{dt} = \frac{e}{c}\mathbf{v} \times \mathbf{B} \quad , \quad \frac{d\mathcal{E}}{dt} = 0 . \tag{11.6.16}$$

Thus, the energy does not change with time, i.e. velocity is constant, and so is $\gamma = 1/\sqrt{1 - v^2/c^2}$. We can use $\mathbf{p} = \gamma m \mathbf{v}$ and write the momentum equation as

$$\frac{d\mathbf{v}}{dt} = \mathbf{v} \times \boldsymbol{\omega}_B , \tag{11.6.17}$$

where

$$\boldsymbol{\omega}_B = \frac{e\mathbf{B}}{\gamma mc} \tag{11.6.18}$$

is the gyration or precession frequency. For small velocities, $\gamma \approx 1$ and $\omega = eB/mc$.
Taking the magnetic field in z-direction, we write the equation in components:

$$\dot{v}_x = \omega v_y \quad , \quad \dot{v}_y = -\omega v_x \quad , \quad \dot{v}_z = 0 \ . \tag{11.6.19}$$

Thus, $v_z = \text{const} = v_\|$.
For motion in the plane normal to the field, introducing a complex combination $V = v_x + iv_y$, we can write the coupled x, y equations as

$$\dot{V} = -i\omega V \ .$$

Its solution is

$$V(t) = V(0)\, e^{-i\omega t} \ , \tag{11.6.20}$$

where $V(0) = v_{x0} + iv_{y0}$. Taking initial velocity in x-direction, i.e. $v_{x0} = v$, $v_{y0} = 0$ gives

$$v_x = v\cos(\omega t) \quad , \quad v_y = -v\sin(\omega t) \ . \tag{11.6.21}$$

The solution for coordinates is

$$R(t) \equiv x(t) + iy(t) = R(0) + i\,\frac{v}{\omega}\, e^{-i\omega t} \ , \tag{11.6.22}$$

or

$$x(t) = x_0 + a\ \sin(\omega t) \ , \tag{11.6.23}$$

$$y(t) = y_0 + a\ \cos(\omega t) \ , \tag{11.6.24}$$

where $a = v/\omega$ is gyration radius. Thus, we deal with a circular motion in the plane perpendicular to $\mathbf{B}$. The rotation is counterclockwise (for a positive charge) when viewed in the direction of magnetic field. Combining it with a uniform motion in the direction parallel to $\mathbf{B}$ results in a helix trajectory with radius a and pitch angle $\alpha = \tan^{-1}(v_\|/\omega_B a)$. The magnitude of a is determined by B and transverse momentum of the particle: $cp_\perp = eBa$.

11.6.3 Motion in combined uniform, static electric and magnetic fields

Consider a situation when $\mathbf{E}$ and $\mathbf{B}$ are not parallel and, furthermore, take the simplest case when they are perpendicular. The equation is still complicated, since the energy is not conserved. In attempt to simplify the treatment, let us switch to a coordinate frame K' moving with velocity $\mathbf{u}$ with respect to the original frame K. Then, in K'-frame

$$\frac{d\mathbf{p}'}{dt} = e\left[\mathbf{E}' + \frac{\mathbf{v}'}{c} \times \mathbf{B}'\right] , \tag{11.6.25}$$

where

$$\mathbf{E}' = \gamma\left(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E}) \tag{11.6.26}$$

$$\mathbf{B}' = \gamma\left(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B}) \tag{11.6.27}$$

with $\boldsymbol{\beta} = \mathbf{u}/c$. The terms $\boldsymbol{\beta}\cdot\mathbf{E}$ and $\boldsymbol{\beta}\cdot\mathbf{B}$ will be eliminated if $\mathbf{u}$ is chosen to be orthogonal to both $\mathbf{E}$ and $\mathbf{B}$. Suppose that $|\mathbf{E}| < |\mathbf{B}|$. Then we take

$$\mathbf{u} = c\,\frac{\mathbf{E}\times\mathbf{B}}{|\mathbf{B}|^2} \quad \text{or} \quad \boldsymbol{\beta} = \frac{\mathbf{E}\times\mathbf{B}}{|\mathbf{B}|^2} \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1-|\mathbf{E}|^2/|\mathbf{B}|^2}} . \tag{11.6.28}$$

As a result

$$\mathbf{E}' = \gamma\left(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}\right) = \gamma\left(\mathbf{E} - \frac{\mathbf{B}\times(\mathbf{E}\times\mathbf{B})}{|\mathbf{B}|^2}\right) = \gamma\left(\mathbf{E} - \frac{\mathbf{E}|\mathbf{B}|^2 - \mathbf{B}(\mathbf{E}\cdot\mathbf{B})}{|\mathbf{B}|^2}\right)$$
$$= \gamma\,\mathbf{B}\,\frac{(\mathbf{E}\cdot\mathbf{B})}{|\mathbf{B}|^2} = 0 . \tag{11.6.29}$$

At the last step we used $(\mathbf{E}\cdot\mathbf{B}) = 0$. For the magnetic field, we obtain

$$\mathbf{B}' = \gamma\left(\mathbf{B} - \boldsymbol{\beta}\times\mathbf{E}\right) = \gamma\left(\mathbf{B} + \frac{\mathbf{E}\times(\mathbf{E}\times\mathbf{B})}{|\mathbf{B}|^2}\right) = \gamma\left(\mathbf{B} + \frac{\mathbf{E}(\mathbf{E}\cdot\mathbf{B}) - \mathbf{B}|\mathbf{E}|^2}{|\mathbf{B}|^2}\right)$$
$$= \gamma\,\mathbf{B}\left(1 - \frac{|\mathbf{E}|^2}{|\mathbf{B}|^2}\right) = \gamma\,\mathbf{B}\,\frac{1}{\gamma^2} = \frac{1}{\gamma}\mathbf{B} \tag{11.6.30}$$

So, in the frame K' the only field acting on the particle is a static magnetic field which points in the same direction as $\mathbf{B}$, but weaker than $\mathbf{B}$ due to the $1/\gamma$ factor. Thus the motion of the particle in the K' frame is spiraling around the line of (magnetic) force.

As viewed from the original system K, the gyration is accompanied by a uniform drift in the direction $\mathbf{u}$ perpendicular to $\mathbf{E}$ and $\mathbf{B}$. The direction of the drift does not depend on the charge of the particle.

If $|\mathbf{E}| > |\mathbf{B}|$, the electric field is so strong that the particle is continually accelerated in the direction of $\mathbf{E}$.

Consider a Lorentz transformation to the frame moving with velocity $\mathbf{u}' = c(\mathbf{E} \times \mathbf{B})/|\mathbf{E}|^2$ corresponding to $\gamma' = \sqrt{1 - |\mathbf{B}|^2/|\mathbf{E}|^2}$. In this frame

$$\begin{aligned}\mathbf{E}' = \gamma'\left(\mathbf{E} + \boldsymbol{\beta}' \times \mathbf{B}\right) =& \gamma'\left(\mathbf{E} - \frac{\mathbf{B} \times (\mathbf{E} \times \mathbf{B})}{|\mathbf{E}|^2}\right) = \gamma'\left(\mathbf{E} - \frac{\mathbf{E}|\mathbf{B}|^2 - \mathbf{B}(\mathbf{E} \cdot \mathbf{B})}{|\mathbf{E}|^2}\right) \\ &= \gamma'\,\mathbf{E}\,\frac{|\mathbf{E}|^2 - |\mathbf{B}|^2}{|\mathbf{E}|^2} = \gamma'\,\mathbf{E}\left(1 - \frac{|\mathbf{B}|^2}{|\mathbf{E}|^2}\right) = \frac{1}{\gamma'}\,\mathbf{E} \qquad (11.6.31)\end{aligned}$$

and

$$\begin{aligned}\mathbf{B}' =& \gamma'\left(\mathbf{B} - \frac{\mathbf{u}' \times \mathbf{E}}{c}\right) = \gamma'\left(\mathbf{B} + \frac{\mathbf{E} \times (\mathbf{E} \times \mathbf{B})}{|\mathbf{E}|^2}\right) \\ &= \gamma'\left(\mathbf{B} + \frac{\mathbf{E}(\mathbf{E} \cdot \mathbf{B}) - \mathbf{B}|\mathbf{E}|^2}{|\mathbf{E}|^2}\right) = 0\ . \qquad (11.6.32)\end{aligned}$$

Thus, we have only electric field in frame K', and the particle is continually accelerated in the direction of $\mathbf{E}$.