804 Final Exam (40 points). 05/01/07, 15:45 ÷ 18:45 p.m.

Problem 1.

Consider the vector potential $A(\vec{r}) = \vec{r} \times (\vec{r} \times \hat{e}_3)$.

(a) What is the magnetic field?

(b) Does this potential satisfy Coulomb gauge condition $\vec{\nabla} \cdot \vec{A} = 0$? If not, modify $\vec{A}$ such that the magnetic field remains the same but the Coulomb condition is satisfied.

Solution

$$\vec{A} \;=\; z\vec{r} - r^2\hat{e}_3 \;=\; xz\hat{e}_1 + yz\hat{e}_2 - (x^2+y^2)\hat{e}_3$$

The magnetic field is $\vec{B} = \vec{\nabla} \times \vec{A} = 3x\hat{e}_1 - 3y\hat{e}_2$. (In the spherical polar coordinates $\vec{A} = r^2 \sin\theta\, \hat{\theta}$ and $\vec{B} = -3r\sin\theta\, \hat{\phi}$).

Since $\vec{\nabla} \cdot \vec{A} = 2z$ the potential is not in the Coulomb gauge. To bring it to Coulomb gauge we need a transformation

$$\vec{A}' \to \vec{A} + \vec{\nabla}\Lambda$$

so that

$$\vec{\nabla} \cdot \vec{A}' \;=\; 0 \;\Rightarrow \nabla^2\Lambda \;=\; -\vec{\nabla} \cdot \vec{A} \;=\; -2z$$

The simplest solution of $\nabla^2\Lambda = -2z$ is $\Lambda = -\frac{z^3}{3}$ so we get

$$\vec{A}' \;=\; xz\hat{e}_1 + yz\hat{e}_2 - (x^2+y^2)\hat{e}_3 + \vec{\nabla}\Lambda \;=\; xz\hat{e}_1 + yz\hat{e}_2 - (x^2+y^2+z^2)\hat{e}_3$$

Problem 2

A TEM wave propagates along a coaxial cylindrical wave guide (made from a perfect conductor) with inner radius a and outer radius b. Find the surface current on the outer cylinder. Is there a total current flow along the outer cylinder?

(Reminder/hint: a cylindrically symmetric solution of the 2-dim Laplace eqn is $\vec{E} \sim \hat{s}/s$ for the electric field and $\vec{B} \sim \hat{\phi}/s$ for the magnetic field)

Solution

The magnetic field of the TEM wave has the form

$$\vec{B}(\vec{s}, z, t) \;=\; \frac{C_0}{s}\hat{\phi}e^{ikz-i\omega t}$$

where C_0 is some constant. From the boundary conditions at the surface of a perfect conductor we get

$$\vec{K} \;=\; \hat{n} \times \vec{H} \;=\; -\mu_0\vec{s} \times \vec{B} \;=\; \frac{\mu_0 C_0}{b}e^{ikz-i\omega t}\hat{e}_3$$

so the real surface current is

$$\vec{K}(\vec{s}, z, t) \;=\; \Re\frac{\mu_0 C_0}{b} e^{ikz-i\omega t}\hat{e}_3 \;=\; \frac{\mu_0 C_0}{b}\cos(kz-\omega t)\;\hat{e}_3$$

The current (in the z direction) along the whole perimeter is

$$I \;=\; \int dlK \;=\; 2\pi\mu_0 C_0 \cos(kz-\omega t)$$

$\Rightarrow$ alternating total current along the conductor.

Problem 3 .

A particle of charge q moves in a circle of radius a at a constant angular velocity ω. Assume that the circle lies in the x, y plane, centered at the origin and at time $t = 0$, the charge is at $(a, 0)$ on the positive x axis. For points on the z axis, find

(a) the Lienard-Wiechert potentials and

(b) the time-averaged electric field.

Solution

(a). The Lienard-Wiechert potentials are given by

$$\begin{aligned}\phi(\vec{r}, t) \;&=\; \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{w}(t_r)| - \frac{1}{c}\vec{v}(t_r)\cdot(\vec{r}-\vec{w}(t_r))} \\ \vec{A}(\vec{r}, t) \;&=\; \frac{\vec{v}(t_r)}{c^2}\phi(\vec{r}, t)\end{aligned}$$

In our case $|\vec{r}-\vec{w}(t)| = \sqrt{z^2+a^2}$ and $\vec{v}(t)\cdot(\vec{r}-\vec{w}(t)) = 0$ for any t so we get

$$\begin{aligned}\phi(\vec{r}, t) \;&=\; \frac{q}{4\pi\epsilon_0\sqrt{z^2+a^2}} \\ \vec{A}(\vec{r}, t) \;&=\; \frac{\vec{v}(t_r)}{c^2}\phi(\vec{r}, t) \;=\; \frac{\mu_0 q}{4\pi\sqrt{z^2+a^2}}[-\sin(t-t_r)\hat{e}_1 + \cos(t-t_r)\hat{e}_2]\end{aligned}$$

where $t_r = \frac{1}{c}\sqrt{z^2+a^2}$ is the retarded time.

(b). By symmetry, the time-averaged electric field is collinear to the z axis so it is sufficient to find $E_z(z, 0, 0)$

$$\langle E_z(z,0,0)\rangle \;=\; -\langle\frac{\partial}{\partial z}\phi(z,0,0)\rangle - \langle\frac{\partial A_z}{\partial t}\rangle \;=\; -\frac{\partial}{\partial z}\phi(z,0,0) \;=\; \frac{qz}{4\pi\epsilon_0(z^2+a^2)^{3/2}}$$

so $\langle\vec{E}(z,0,0) \;=\; \frac{qz}{4\pi\epsilon_0(z^2+a^2)^{3/2}}\hat{e}_3$.

Problem 4.

An insulated circular ring of radius b lies in the x, y plane centered at the origin. It carries a linear charge density $\lambda = \lambda_0 \sin\phi$ where ϕ is an azimuthal angle. The ring is set spinning at a constant angular velocity $\vec{\omega} = \omega\hat{z}$. Find the power radiated by the ring.

Solution

The main part of the radiated power comes from the electric dipole radiation. The electric dipole moment of the ring at rest is

$$\vec{p} \ = \ b\int_0^{2\pi} d\phi \ \lambda \sin\phi \ \vec{r} \ = \ b\int_0^{2\pi} d\phi \ \lambda \sin\phi \ (\hat{e}_1 b\cos\phi + \hat{e}_2 b\sin\phi) \ = \ \pi\lambda b^2 \hat{e}_2$$

This dipole (located at the origin) is set spinning in the x, y plane with angular frequency ω so

$$\vec{p}(t) \ = \ \pi\lambda b^2(\hat{e}_2 \cos\omega t - \hat{e}_1 \sin\omega t)$$

From the Larmor formula

$$P = \frac{\mu_0}{6\pi c}(\ddot{\vec{p}})^2 \ = \ \frac{\mu_0\pi}{6c}\lambda^2\omega^4 b^4$$

Problem 5.

In a certain frame K the electric field $\vec{E}$ and the magnetic filed $\vec{B}$ are orthogonal. Is there a frame where the field is

(a) purely electric or (b) purely magnetic,

and with what velocity should that frame(s) move with respect to K?

Solution

In the frame K the second Lorentz invariant $\vec{E}\cdot\vec{B} = 0$ while the first invariant $E^2 - B^2$ is positive when $|\vec{E}| > |\vec{B}|$ and negative for $|\vec{E}| < |\vec{B}|$ so nothing forbids to have $\vec{B}' = 0$ in the former case and $\vec{E}' = 0$ in the latter. To get the velocity of the relevant boost, consider the Lorentz transformations of the electric and magnetic fields

$$\vec{E}' \ = \ \gamma(\vec{E} + \vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E})$$
$$\vec{B}' \ = \ \gamma(\vec{B} - \vec{\beta}\times\vec{E}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B})$$

In the first case (when $E > B$) we want to have $B' = 0$ so

$$(\vec{B} - \vec{\beta}\times\vec{E}) \ = \ \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B})$$

Multiplying both sides by $\vec{\beta}$ we get

$$\vec{\beta}\cdot(\vec{B}-\vec{\beta}\times\vec{E}) \;=\; \frac{\gamma}{\gamma+1}\beta^2(\vec{\beta}\cdot\vec{B}) \;\Rightarrow\; \vec{\beta}\cdot\vec{B}=0 \;\Leftrightarrow\; \vec{v}\perp\vec{B}$$

so $\vec{B}=-\vec{\beta}\times\vec{E}$. The simplest choice is to take $\vec{v}$ orthogonal to both $\vec{E}$ and $\vec{B}$, then

$$v=c\frac{B}{E},\quad \vec{v}\perp\vec{E},\vec{B}$$

Similarly, in the second case we can take $v=c\frac{E}{B},\;\; \vec{v}\perp\vec{E},\vec{B}$ and the resulting $\vec{E}'$ will vanish.

(In the SI units $v=c^2\frac{B}{E}$ and $v=\frac{E}{B}$ for the first and the second case, respectively).