804 Final Exam (40 points). 05/07/08, 2:00 ÷ 5:00 p.m.

Problem 1.

A long cylindrical metal wire of radius a carries a uniform current of density J in the axial direction. The conductivity of the metal is σ. Calculate the magnitude and the direction of Poynting vector at the surface of the wire.

Solution

From Ampere's law

$$\vec{B}(s) \stackrel{s\geq a}{=} \frac{\mu_0 I}{2\pi s}\hat{\phi} \quad \Rightarrow \quad \vec{B}(a) = \frac{\mu_0 I}{2\pi a}\hat{\phi} = \frac{\mu_0}{2}Ja\hat{\phi}$$

From Ohm's law

$$\vec{J} = \sigma\vec{E} \quad \Rightarrow \quad \vec{E} = \frac{1}{\sigma}J\hat{e}_3$$

$\Rightarrow$ the Poynting vector is

$$\vec{S}(a) = \frac{1}{\mu_0}\vec{E}(a)\times\vec{B}(a) = \frac{J^2 a}{2\sigma}\hat{e}_3\times\hat{\phi} = -\frac{J^2 a}{2\sigma}\hat{s}$$

The sign reflects the fact that the energy is pumped *into* the wire (to be dissipated like the ohmic heat).

Problem 2

In a certain gauge the magnetic vector potenial has the form

$$\vec{A}(t,\vec{r}) = -t\vec{E}$$

($\vec{E}$ is constant) and the scalar electic potential vanishes. Find the gauge transformation $\Lambda(t,\vec{r})$ to another gauge where the magnetic vector potential is zero and write down the corresponding electric potential.

Solution

The gauge transformation has the form

$$\vec{A}'(t,\vec{r}) = \vec{A}(t,\vec{r}) + \vec{\nabla}\Lambda(t,\vec{r}), \quad \Phi'(t,\vec{r}) = \Phi(t,\vec{r}) - \frac{\partial}{\partial t}\Lambda(t,\vec{r})$$

From the requirement $\vec{A}'(t,\vec{r}) = 0$ we get

$$\vec{\nabla}\Lambda(t,\vec{r}) = t\vec{E}$$

Let us choose OZ axis in the $\vec{E}$ direction so $\vec{E} = E\hat{e}_3$ so the above equation reduces to

$$\frac{\partial}{\partial z}\Lambda(t,x,y,z) = tE \quad \Rightarrow \quad \Lambda(t,x,y,z) = ztE$$

and the corresponding scalar potential takes the form

$$\Phi'(t,\vec{r}) \;=\; -\frac{\partial}{\partial t}\Lambda(t,\vec{r}) \;=\; -zE$$

which is our usual choice for the potential corresponding to the uniform electric field $E\hat{e}_3$.

Problem 3

A linearly polarized electromagnetic plane wave is incident at angle θ on an infinitely large plane made from a perfect conductor. The electric field is orthogonal to the plane of incidence. Find charge and current densities induced on the conducting plane.

Solution

Inside the perfect conductor electric and magnetic fields vanish, therefore the boundary conditions for the fields right above the conductor's surface are

$$\vec{E}^{\parallel} = 0, \quad E_{\perp} = \frac{\sigma}{\epsilon_0}, \quad B_{\perp} = 0, \quad \vec{B}\times\hat{n} = \mu_0\vec{K}$$

Let us choose the XZ plane of incidence. The wave incident on the XY plane from above has the form ($\vec{k} = k\sin\theta\hat{e}_1 - k\cos\theta\hat{e}_3$)

$$\vec{E}_i = E_i\hat{e}_2 e^{ikx\sin\theta - ikz\cos\theta}$$

The reflected wave has momentum $\vec{k}_r = k\sin\theta\hat{e}_1 + k\cos\theta\hat{e}_3$:

$$\vec{E}_r = -E_i\hat{e}_2 e^{ikx\sin\theta + ikz\cos\theta}$$

The opposite sign follows from the boundary condition at the conductor's surface - see below. The sum of the incident and reflected waves is

$$\vec{E} = \; -2iE_i\hat{e}_2 e^{ikx\sin\theta}\sin kz\cos\theta$$

which satisfies the boundary condition $\vec{E}^{\parallel}\Big|_{z=0} = 0$. From the second equation

$$\sigma = \epsilon_0\vec{E}\cdot\hat{e}_3\Big|_{z=0} = 0$$

The magnetic field for the incident wave is $\vec{B} = \frac{\hat{k}}{c}\times\vec{E}_i \;=\; \frac{k}{c}(\sin\theta\hat{e}_1 - \cos\theta\hat{e}_3)\times\vec{E}_i \quad\Rightarrow$

$$\vec{B}_i = \frac{k}{c}(\hat{e}_1\cos\theta + \hat{e}_3\sin\theta)E_i e^{ikx\sin\theta - ikz\cos\theta}$$

Similarly, for the reflected wave $\vec{B} = \frac{\hat{k}_r}{c} \times \vec{E}_r \;=\; \frac{k}{c}(\sin\theta\hat{e}_1 + \cos\theta\hat{e}_3) \times \vec{E}_r \;\Rightarrow$

$$\vec{B}_r = \frac{k}{c}E_i(\hat{e}_1\cos\theta - \hat{e}_3\sin\theta)e^{ikx\sin\theta+ikz\cos\theta}\hat{e}_2$$

and therefore the total magnetic field (above the plane) is

$$\vec{B} = 2\frac{k}{c}E_i e^{ikx\sin\theta}\Big[\hat{e}_1\cos\theta\cos(kz\cos\theta) - i\hat{e}_3\sin\theta\sin(kz\cos\theta)\Big]$$

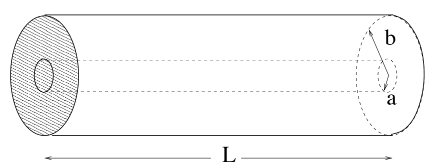
and therefore

$$\vec{K} = \Re\,\frac{1}{\mu_0}\vec{B}\times\hat{e}_3\Big|_{z=0} = -2\frac{k}{\mu_0 c}\cos\theta E_i\cos(kx\sin\theta)\hat{e}_2$$

Problem 4. The co-axial wave guide is closed at both ends by metal lids so the space between two cylinders makes a resonant cavity. Show that this cavity can support a TEM wave ($E_z = B_z = 0$) and find:

(a) Allowed frequencies

(b) Electric and magnetic fields in the TEM wave



Solution

The general form of the TEM wave is $\vec{E}_{\text{TEM}} = \vec{E}_T(x,y)e^{\pm ikz - i\omega t}$ where $k = \frac{\omega}{c}$ as for the plane wave. The function $\vec{E}_T(x,y)$ satisfies the equations

$$\nabla\times\vec{E}_T = 0, \quad \nabla\cdot\vec{E}_T = 0$$

In accordance with the first of the above equations we can describe E_T by a scalar potential

$$\vec{E}_T(x,y) = -\nabla_T\phi(x,y)$$

and from the second equation we obtain

$$\nabla_T^2\phi(x,y) = \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)\phi(x,y) = 0$$

which is the Laplace equation for the two-dimensional electrostatics. It is convenient to solve it in the polar coordinates s, φ

$$\nabla_T^2 \phi(x,y) = \Big(\frac{1}{s}\frac{\partial}{\partial s} s \frac{\partial}{\partial s} + \frac{1}{s^2}\frac{\partial^2}{\partial \varphi^2}\Big)\phi(s,\varphi) = 0$$

Let us choose the potential of the inner cylinder (of radius a) to be zero and let us denote the potential of the outer cylinder (of radius b) by V, then the boundary conditions for the above Laplace equation are $\phi(a,\varphi) = 0$ and $\phi(b,\varphi) = V$. By cylindrical symmetry, the solution of the Laplace equation depends only on s and we easily obtain

$$\frac{\partial}{\partial s} s \frac{\partial \phi(s)}{\partial s} = 0 \quad \Rightarrow \quad s\frac{\partial \phi(s)}{\partial s} = A \quad \Rightarrow \quad \phi(s) = A \ln s + B$$

Taking into account the boundary conditions we get

$$\phi(s) = V\frac{\ln s/a}{\ln b/a} \quad \Rightarrow \quad \vec{E}_T(s,\varphi) = -\hat{s}\frac{\partial \phi}{\partial s} = -\frac{V\hat{s}}{s \ln b/a}$$

(cf. the problem of capacitance of the coaxial cable) and therefore

$$\vec{E}_{\text{TEM}} = -\frac{V\hat{s}}{s\ln b/a} e^{\pm ikz - i\omega t}$$

We have found the solution of the Laplace equation satisfying the boundary conditions at the surface of the inner and outer cylinders. To satisfy the boundary condition $\hat{n} \times \vec{E}_{\text{TEM}} = \hat{e}_3 \times \vec{E}_T = 0$ at $z = 0, L$ we need the standing waves

$$\vec{E}_{\text{TEM}} = \frac{V\hat{s}}{s\ln b/a} e^{-i\omega_n t} \sin k_n z$$

where $k_n = \frac{\pi n}{L}$ as usual. Finally, we obtain

$$\vec{E}_{\text{TEM}} = \frac{V\hat{s}}{s\ln b/a} e^{-i\frac{\pi n}{L}ct} \sin \frac{\pi n}{L} z$$
$$\vec{B}_{\text{TEM}} = \frac{i}{\omega}\vec{\nabla} \times \vec{E}_{\text{TEM}} = -\frac{i}{c}\frac{V\hat{\varphi}}{s\ln b/a} e^{-i\frac{\pi n}{L}ct} \cos \frac{\pi n}{L} z$$

It is easy to see that these fields satisfy boundary conditions $\hat{n} \times \vec{E} = 0$ and $\hat{n} \cdot \vec{B} = 0$ at all relevant surfaces. The allowe frequencies are $\omega_n = \frac{\pi n}{L} c$.

Problem 5.

An insulated circular ring of radius b lies in the x, y plane centered at the origin. It carries a linear charge density $\lambda = \lambda_0 \sin \phi$ where ϕ is an azimuthal angle. The ring is set spinning at a constant angular velocity $\vec{\omega} = \omega \hat{z}$. Find the power radiated by the ring.

Solution

The (electric) dipole moment of the ring is

$$\vec{p} \;=\; \oint dl\; \vec{r}\lambda(\vec{r}) \;=\; b\int_0^{2\pi} d\varphi\; b(\hat{e}_1\cos\phi + \hat{e}_2\sin\phi)\lambda_0\sin\varphi \;=\; b^2\pi\lambda_0\hat{e}_2$$

Thus, the dipole moment $p_0 = \pi b^2\lambda_0$ is set spinning around the axis orthogonal to the direction of the dipole:

$$\vec{p}(t) = p_0\hat{e}_1\sin\omega t + p_0\hat{e}_2\cos\omega t$$

By Larmor formula

$$P = \frac{\mu}{6\pi c}|\ddot{\vec{p}}|^2 \;=\; \frac{\mu\omega^4 p_0^2}{6\pi c} \;=\; \frac{\mu\pi\omega^4\lambda_0^2 b^4}{6c}$$

Problem 5.

Is it possible for $\vec{E}$ and $\vec{B}$ at some point to be parallel in one frame and antiparallel in some other frame?

Solution

No. The scalar product of $\vec{E}$ and $\vec{B}$ is a relativistic invariant ($F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\vec{E}\cdot\vec{B}$) and therefore it cannot change the sign by going to another frame.