Solution

a) Find the difference $c - v \equiv \delta v$ between the velocity v of such a proton and the speed of light $c \approx 3 \times 10^8$m/sec.

$$\frac{1}{\gamma^2} = 1 - \beta^2 = (1-\beta)(1+\beta) \approx 2(1-\beta)$$

Hence,

$$\delta v = c(1-\beta) \approx \frac{c}{2\gamma^2} = \frac{3 \times 10^8}{2 \times 49 \times 10^6}\,\text{m/sec} \approx 3\,\text{m/sec}\ .$$

In fact, LHC is a collider, in which two protons having this energy in the laboratory frame move towards each other (along, say, x axis).

b) Take the frame in which one of the protons is at rest. What is the velocity v_2 of the second proton in that frame?

$$v_2 = \frac{2v}{1 + v^2/c^2} = c\frac{2\beta}{1+\beta^2}$$

Since v_2 is very close to the speed of light, represent it as $v_2 = c - \delta v_2$, and find δv_2.

$$1 - \beta_2 = 1 - \frac{2\beta}{1+\beta^2} = \frac{(1-\beta)^2}{1+\beta^2} \approx \frac{1}{2}(1-\beta)^2 = \frac{1}{8\gamma^4} = \frac{1}{2}\frac{\delta v^2}{c^2}\ ,$$

and

$$\delta v_2 = c(1-\beta_2) \approx \frac{1}{2}\frac{\delta v^2}{c^2} = \frac{1}{2}\delta v\frac{\delta v}{c} \approx 1.5 \times 10^{-8}\,\text{m/sec}\ .$$

c) What is the energy of the second proton in the rest frame of the first one?

$$\gamma_2 = \frac{1}{\sqrt{1-\beta_2^2}} \approx \frac{1}{\sqrt{2(1-\beta_2)}} \approx \frac{1}{1-\beta} \approx 2\gamma^2\ .$$

Hence,

$$E_2 = m_p c^2\,\gamma_2 \approx 2\gamma^2\,m_p c^2 \approx 10^8\,\text{GeV} = 10^5\,\text{TeV}\ .$$

Or calculating $(p_1 p_2)$ in first (c.m.) frame $(p_1 p_2) = E^2 + p^2 \approx 2E^2 = 2m_p^2\gamma^2$, and then in the second frame $(p_1 p_2) = m_p E_p = \gamma_2 m_p^2$. Hence, $\gamma_2 \approx 2\gamma^2$, and $E_p = 2\gamma^2 m_p^2$

d) Imagine that we are in a rocket that leaves the Earth with the speed v equal to the lab frame speed of the LHC protons. How far from the Earth (in light-years) would we find ourselves after spending 1 year of our life on such a rocket?

$$T = \gamma\tau = 7{,}000\,\text{yr}$$

and

$$L \approx cT \approx 7{,}000\,\text{lt} - \text{yr}\ .$$