Phys. 804 HW 8 Assignment.

Problem

Find power radiated by pure electric dipole $\vec{p}$ rotating in XY plane with angular velocity ω.

Solution #1

The rotating dipole can be represented by the dipole moment

$$\vec{p}(t) \;=\; p[\hat{e}_1\cos(\omega t)+\hat{e}_2\sin(\omega t)] \;=\; \Re p(\hat{e}_1+i\hat{e}_2)e^{-i\omega t}$$

The electric and magnetic fields of the oscillating dipole $\vec{p}(t)=\vec{p}e^{-i\omega t}$ are obtained in Lect. 12

$$\begin{aligned}\vec{B}(r,t) &= \mu_0\frac{ck^2}{4\pi r}\hat{n}\times\vec{p}\,\frac{e^{ikr}}{r} \;=\; \hat{n}\times(\hat{e}_1+i\hat{e}_2)\frac{\mu_0 p\omega^2}{4\pi cr}e^{ikr-i\omega t}\\ \vec{E}(r,t) &= -c\hat{n}\times\vec{B} \;=\; \;=\; -\frac{\mu_0 p\omega^2}{4\pi cr}\hat{n}\times(\hat{n}\times(\hat{e}_1+i\hat{e}_2))e^{ikr-i\omega t}\end{aligned}$$

The time-averaged Poynting vector is

$$\begin{aligned}\langle\vec{S}\rangle &= \frac{1}{2}\Re\vec{E}\times\vec{H}^* \;=\; -\frac{c}{2\mu_0}(\hat{n}\times\vec{B})\times\vec{B}^* \;=\; \frac{c}{2\mu_0}\hat{n}|\vec{B}|^2\\ &= \frac{\mu_0 p^2\omega^4}{32\pi^2cr^2}(\hat{n}\times\hat{e}_1+i\hat{n}\times\hat{e}_2)\cdot(\hat{n}\times\hat{e}_1-i\hat{n}\times\hat{e}_2)\\ &= \frac{\mu_0 p^2\omega^4}{32\pi^2cr^2}(|\hat{n}\times\hat{e}_1|^2+|\hat{n}\times\hat{e}_2|^2) \;=\; \frac{\mu_0\omega^4}{32\pi^2cr^2}(1+\cos^2\theta)\end{aligned}$$

since $\hat{n}=\hat{r} \;=\; \sin\theta(\cos\phi\hat{e}_1+\sin\phi\hat{e}_2)+\cos\theta\hat{e}_3$.
The radiated power is

$$P \;=\; r^2\int d\Omega\;\langle\vec{S}\rangle\cdot\hat{r} \;=\; \frac{\mu_0 p^2\omega^4}{6\pi c}$$

Solution #2

From symmetry it is clear that radiated power does not depend on time, so we can use formula

$$P \;=\; \frac{\mu_0}{6\pi c}|\ddot{\vec{p}}|^2$$

at $t=0$. Since $\vec{p}(\vec{r},t) \;=\; p[\hat{e}_1\cos(\omega t)+\hat{e}_2\sin(\omega t)]$ we get

$$\ddot{\vec{p}}(t=0) \;=\; -p\omega^2\hat{e}_1 \quad\Rightarrow\quad |\ddot{\vec{p}}(t=0)|^2 \;=\; p^2\omega^4$$

and the radiated power is

$$P \;=\; \frac{\mu_0}{6\pi c}p^2\omega^4$$