Problem 1 .

An iron sphere of radius R carries a charge Q and has a uniform magnetization $\vec{M} = M\hat{e}_3$. It is initially at rest. Find

(a) Angular momentum stored in the fields

(b) If the sphere is demagnetized by heating (keeping $\vec{M}$ uniform), by use of Faraday's law find the induced electric field, then find the torque induced by $\vec{E}$ on the sphere, and finally the angular momentum imparted to sphere as M goes to zero.

Hint: The magnetic field outside the sphere is equal to the magnetic field of a pure dipole with $\vec{m} = \frac{4}{3}\pi R^3 \vec{M}$

Solution

(a)

The electric and magnetic fields (at $r > R$) are:

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}, \quad \vec{B} = \frac{\mu_0}{4\pi r^3}(3\hat{r}(\vec{m}\cdot\hat{r}) - \vec{m}) \quad \Rightarrow \quad \vec{E}\times\vec{B} \;=\; \frac{Q\mu_0 m}{16\pi^2\epsilon_0 r^5}\hat{e}_3\times\vec{r}$$

where $\vec{m} = \frac{4}{3}\pi R^3 \vec{M}$.

The angular momentum stored in the fields is

$$\int_R^\infty drr^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \epsilon_0 \vec{r}\times(\vec{E}\times\vec{B}) = \int_R^\infty drr^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \; \frac{\mu_0 m Q}{16\pi^2 r^4}(\hat{e}_3 - (\hat{e}_3\cdot\hat{r})\hat{r})$$

By symmetry, the angular momentum $\vec{L}$ is collinear to $\hat{e}_3$ so

$$\vec{L} \;=\; \hat{e}_3 \int_R^\infty drr^2 \int_0^\pi d\theta \int_0^{2\pi} d\phi \; \frac{\mu_0 m Q}{16\pi^2 r^4}\sin^3\theta \;=\; \frac{\mu_0 Q}{6\pi R}\vec{m} \;=\; \frac{2}{9}\mu_0 Q R^2 M$$

(b)

If $\vec{m} = m(t)\hat{e}_3$ the induced electric field takes the form

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\partial}{\partial t}\frac{\mu_0 m(t)\hat{e}_3\times\hat{r}}{4\pi r^2} = -\frac{\mu_0}{4\pi r^2}\hat{e}_3\times\hat{r}\frac{dm}{dt} = -\frac{\mu_0}{4\pi r^2}\hat{\phi}\sin\theta\frac{dm}{dt}$$

The torque for the charge dq on the surface is

$$d\vec{\tau} = dq\;\vec{r}\times\vec{E} = dq\frac{\mu_0}{4\pi R}\hat{\theta}\sin\theta\frac{dm}{dt}, \qquad\qquad \hat{r}\times\hat{\phi} = -\hat{\theta}$$

Again, from symmetry we know that $\vec{\tau} \parallel \hat{e}_3$ so

$$d\tau_3 \;=\; -\,dq\frac{\mu_0}{4\pi R}\sin^2\theta\frac{dm}{dt}$$

Since the charge is distributed uniformly over the surface of the sphere $dq = \frac{Q}{4\pi R^2} R^2 \sin\theta d\theta d\phi$ and the total torque is

$$\tau_3 \;=\; -\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \;\sin^2\theta \frac{\mu_0 Q}{16\pi^2 R}\frac{dm}{dt} \;=\; -\frac{\mu_0 Q}{6\pi R}\frac{dm}{dt}$$

The angular momentum imparted to sphere is

$$\vec{m} \;=\; \hat{e}_3\int_0^\infty \tau_3(t)dt = -\frac{\mu_0 Q}{6\pi R}\hat{e}_3\int_0^\infty \frac{dm}{dt}dt = \frac{\mu_0 Q}{6\pi R}m\hat{e}_3 = \frac{2}{9}\mu_0 Q R^2 M\hat{e}_3$$

Problem 2.

A circular wire of radius R lies in x, y plane with the center at the origin. It carries current

$$I(t) \;=\; I_0\theta(t)$$

Find the magnetic field at $z = 0$.

Solution

W.l.o.g., consider the point of observation (x,0,0).

$$\begin{aligned}
d\vec{B} \;&=\; \frac{\mu_0}{4\pi}Id\vec{l}\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\theta(c^2t^2-|\vec{r}-\vec{r}'|^2)\\
&= \frac{\mu_0 I}{4\pi}(-\hat{e}_1\sin\phi+\hat{e}_2\cos\phi)\times((x-R\cos\phi)\hat{e}_1 - R\sin\phi\hat{e}_2)\frac{\theta(c^2t^2-x^2-R^2+2xR\cos\phi)}{|x^2+R^2-2xR\cos\phi|^{3/2}}\\
&= \frac{\mu_0 I}{4\pi}\hat{e}_3(R-x\cos\phi)\frac{\theta(c^2t^2-x^2-R^2+2xR\cos\phi)}{|x^2+R^2-2xR\cos\phi|^{3/2}}
\end{aligned}$$

$$\begin{aligned}
\vec{B} \;&=\; \int d\vec{B} \;=\; \frac{\mu_0 IR}{4\pi}\hat{e}_3\int_0^{2\pi} d\phi\frac{\theta(c^2t^2-x^2-R^2+2xR\cos\phi)}{|x^2+R^2-2xR\cos\phi|^{3/2}}(R-x\cos\phi)\\
&= \frac{\mu_0 IR}{\pi}\hat{e}_3\int_0^{2\pi} d\alpha\;\frac{\theta(c^2t^2-x^2-R^2+2xR\cos 2\alpha)}{|x^2+R^2-2xR\cos 2\alpha|^{3/2}}(R-x\cos 2\alpha)\\
&= \frac{\mu_0 IR}{\pi}\hat{e}_3\int_0^{2\pi} d\alpha\;\frac{\theta(c^2t^2-(x-R)^2-2xR\sin^2\alpha)}{|(x-R)^2+2xR\sin^2\alpha|^{3/2}}(R-x+2x\sin^2\alpha)\\
&= \frac{4\mu_0 IR}{\pi}\hat{e}_3\int_0^{\pi/2} d\alpha\;\frac{\theta(c^2t^2-(x-R)^2-2xR\sin^2\alpha)}{|(x-R)^2+2xR\sin^2\alpha|^{3/2}}(R-x+2x\sin^2\alpha)\\
&= \frac{4\mu_0 IR}{\pi}\hat{e}_3\int_0^1 \frac{d\xi}{\sqrt{1-\xi^2}}\frac{\theta(c^2t^2-(x-R)^2-2xR\xi^2)}{|(x-R)^2+2xR\xi^2|^{3/2}}(R-x+2x\xi^2)\\
&= \frac{4\mu_0 IR}{\pi}\hat{e}_3\theta\Big(t-\frac{|x-R|}{c}\Big)\int_0^\kappa \frac{d\xi}{\sqrt{1-\xi^2}}\frac{R-x+2x\xi^2}{|(x-R)^2+2xR\xi^2|^{3/2}}
\end{aligned}$$

where $\kappa \equiv \min\Big(\frac{\sqrt{c^2t^2-(x-R)^2}}{2xR}, 1\Big)$.

This integral can be performed only numerically. Two simple cases are $x \gg R$ and $x = 0$:

Large $x \gg R$

$$\vec{B} \;=\; \frac{\mu_0 IR}{4\pi x^3}\hat{e}_3\theta\Big(t - \frac{x}{c}\Big)\int_0^{2\pi} d\phi\Big(1 + 3\frac{R}{x}\cos\phi\Big)(R - x\cos\phi) \;=\; -\,\frac{\mu_0\vec{m}}{4\pi x^3}\theta\Big(t - \frac{x}{c}\Big)$$

where $\vec{m} = \pi IR^2\hat{e}_3$. This agrees with the formula for the field of a pure dipole $\vec{m} = \pi R^2 I\hat{e}_3$.

At $x = 0$

$$\vec{B} \;=\; \frac{\mu_0 IR}{4\pi}\hat{e}_3\theta\Big(t - \frac{x}{c}\Big)\int_0^{2\pi} d\phi\frac{1}{R^2} \;=\; \frac{\mu_0 I}{2R}\theta\Big(t - \frac{x}{c}\Big)$$

Problem 3

A circularly polarized electromagnetic plane wave is normally incident on an infinitely large plane made from a perfect conductor. Find the charge and current densities induced on the conducting plane.

Solution

Let us choose the z axis in the direction normal to the plane and the wave coming from above the plane. The incident wave has the form ($\hat{e}_3 \times \hat{e}_+ = -i\hat{e}_+$)

$$\vec{E} \;=\; -\,i\hat{e}_+E_0e^{-i\omega t + ikz}, \quad \vec{B} \;=\; \frac{\hat{n}}{c}\times\vec{E} \;=\; -\,\frac{\hat{e}_3}{c}\times\vec{E} \;=\; \hat{e}_+\frac{E_0}{c}e^{-i\omega t + ikz}$$

where $\hat{e}_+ = \frac{1}{\sqrt{2}}(\hat{e}_1 + i\hat{e}_2)$ and $-i$ is added for convenience (E_0 is real). The reflected wave is

$$\vec{E} \;=\; \vec{E}_Re^{-i\omega t - ikz}, \quad \vec{B} \;=\; \frac{\hat{n}}{c}\times\vec{E} \;=\; \frac{\hat{e}_3}{c}\times\vec{E}_R\;e^{-i\omega t - ikz}$$

The boundary condition for the electric field is $E_\parallel^{\text{above}} = E_\parallel^{\text{below}} = 0$ so $\vec{E}_R = i\hat{e}_+E_0$ and the reflected wave is

$$\vec{E} \;=\; i\hat{e}_+E_0e^{-i\omega t - ikz}, \qquad \vec{B} \;=\; \frac{\hat{e}_3}{c}\times\vec{E}_R\;e^{-i\omega t - ikz} \;=\; \hat{e}_+\frac{E_0}{c}e^{-i\omega t - ikz}$$

The sum of the reflected and incident waves takes the form:

$$\vec{E} \;=\; 2\hat{e}_+E_0\sin kz\;e^{-i\omega t}, \quad \vec{B} \;=\; 2\hat{e}_+\frac{E_0}{c}\cos kz\;e^{-i\omega t}$$

Now, σ and K can be found from the remaining boundary conditions

$$\epsilon_1\vec{E}_\perp^{\text{above}} - \epsilon_2\vec{E}_\perp^{\text{below}} = \sigma, \quad \frac{1}{\mu_1}\vec{B}_\perp^{\text{above}} - \frac{1}{\mu_2}\vec{B}_\perp^{\text{below}} = \vec{K}$$

We get $\sigma = 0$ and $\vec{K} \;=\; \frac{2}{\mu_0 c}\hat{e}_+E_0\;e^{-i\omega t}$. Taking the real part we get

$$K_x \;=\; \frac{2}{\mu_0 c}E_0\;\cos\omega t, \quad K_y \;=\; \frac{2}{\mu_0 c}E_0\;\sin\omega t$$