Problem 1.

A $TE_{1,1}$ wave propagates along the cylindrical wave guide made from a perfect conductor of radius R.

(a) Find the (surface) currents on the surface of the guide.

(b) Is there a total current flow along the wave guide?

Solution

The solution for H_z is

$$H_z = \psi(s,\varphi)e^{ikz}, \qquad \psi(s,\varphi) = J_1(\gamma s)e^{i\varphi}$$

where

$$\gamma R = x'_{11},$$

where $x'_{11} = 1.841...$ is the first root of $J'_1(x) = 0$.

In cylindrical coordinates $\vec{\nabla}_T\psi(s,\varphi) \;=\; \hat{e}_s\frac{\partial\psi}{\partial s} + \hat{e}_\phi\frac{1}{s}\frac{\partial\psi}{\partial\phi}$ so

$$\begin{aligned}
\vec{H}_T \;&=\; \frac{ik}{\gamma^2}e^{ikz}\vec{\nabla}_T\psi \;=\; \frac{ik}{\gamma^2}e^{ikz+i\phi}\Big[\hat{e}_s\gamma J'(\gamma s) + \frac{i}{s}\hat{e}_\phi J_1(\gamma s)\Big]\\
\vec{E}_T \;&=\; -\frac{i\omega\mu}{\gamma^2}e^{ikz}\,\hat{e}_z\times\nabla_T\psi \;=\; -\frac{i\omega\mu}{\gamma^2}e^{ikz+i\phi}\Big[\hat{e}_\phi\gamma J'_1(\gamma s) - \frac{i}{s}\hat{e}_s J_1(\gamma s)\Big]
\end{aligned}$$

The surface current is

$$\vec{K}(\varphi) \;=\; \Re\hat{n}\times\vec{H} \;=\; \Re\hat{s}\times\vec{H} \;=\; -\Re[\hat{e}_\phi + \frac{kR}{{x'_{11}}^2}\hat{e}_z\Big]J_1(x'_{11})e^{i\varphi} \;=\; -[\hat{e}_\phi + \frac{kR}{{x'_{11}}^2}\hat{e}_z\Big]J_1(x'_{11})\cos\varphi$$

The total current along the guide vanishes

$$\int_0^{2\pi}d\varphi K_z(\varphi) \;=\; 0$$

Problem 2.

A particle of charge q moves in a circle of radius a at a constant angular velocity ω. Assume that the circle lies in the x, y plane, centered at the origin and at time $t = 0$, the charge is at $(a, 0)$ on the positive x axis. For points on the z axis, find

(a) the Lienard-Wiechert potentials and

(b) the time-averaged electric field.

Solution

(a). The Lienard-Wiechert potentials are given by

$$\phi(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{w}(t_r)| - \frac{1}{c}\vec{v}(t_r) \cdot (\vec{r} - \vec{w}(t_r))}$$

$$\vec{A}(\vec{r}, t) = \frac{\vec{v}(t_r)}{c^2}\phi(\vec{r}, t)$$

In our case $|\vec{r} - \vec{w}(t)| = \sqrt{z^2 + a^2}$ and $\vec{v}(t) \cdot (\vec{r} - \vec{w}(t)) = 0$ for any t so we get

$$\phi(\vec{r}, t) = \frac{q}{4\pi\epsilon_0\sqrt{z^2 + a^2}}$$

$$\vec{A}(\vec{r}, t) = \frac{\vec{v}(t_r)}{c^2}\phi(\vec{r}, t) \;=\; \frac{\mu_0 q}{4\pi\sqrt{z^2 + a^2}}[-\sin(t - t_r)\hat{e}_1 + \cos(t - t_r)\hat{e}_2]$$

where $t_r = \frac{1}{c}\sqrt{z^2 + a^2}$ is the retarded time.

(b). By symmetry, the time-averaged electric field is collinear to the z axis so it is sufficient to find $E_z(z, 0, 0)$

$$\langle E_z(z, 0, 0)\rangle \;=\; -\langle \frac{\partial}{\partial z}\phi(z, 0, 0)\rangle - \langle \frac{\partial A_z}{\partial t}\rangle \;=\; -\frac{\partial}{\partial z}\phi(z, 0, 0) \;=\; \frac{qz}{4\pi\epsilon_0(z^2 + a^2)^{3/2}}$$

so

$$\langle \vec{E}(z, 0, 0)\rangle \;=\; \frac{qz}{4\pi\epsilon_0(z^2 + a^2)^{3/2}}\hat{e}_3$$

Problem 3.

A particle of charge q and mass m moves through empty space with initial velocity $\vec{v}_0 = v_0\hat{e}_1$ ($v_0 \ll c$) at $t = 0$ in a uniform magnetic field $\vec{B} = B\hat{e}_3$. How long will it take for the particle to lose half of its kinetic energy to radiation? (Assume that the magnetic field is sufficiently weak so that the particle loses half of its energy after many revolutions).

Solution

In the uniform magnetic field, the particle moves around the circle with the radius $R = \frac{v}{\omega_B}$ where $\omega_B = \frac{qB}{m}$ is the cyclotron frequency. The particle will radiate so v (and R) will slowly decrease with time. The radiated power is

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{\mu_0 q^2 \omega_B^2 v^2}{6\pi c} = \frac{\mu_0 q^2 \omega_B^2}{3\pi m c}\ \frac{mv^2}{2}$$

so we have the differential equation

$$P = -\frac{dE_{\mathrm{kin}}(t)}{dt} = -\frac{\mu_0 q^2 \omega_B^2}{3\pi m c} E_{\mathrm{kin}}(t)$$

The solution is

$$E_{\mathrm{kin}}(t) = E_0 e^{-\frac{\mu_0 q^2 \omega_B^2}{3\pi m c} t}$$

so the particle will lose half of its kinetic energy after time

$$t = \frac{3\pi m c}{\mu_0 q^2 \omega_B^2} \ln 2 = \frac{3\pi m^3 c}{\mu_0 q^4 B^2} \ln 2$$