Problem 1.

A TM wave is propagating in the semi-infinite wave guide with perfectly conducting plates located at $x = 0, y > 0$, and $x = a, y > 0$, and $a > x > 0, y = 0$ (see the cross section below).

Find the electric and magnetic fields between the plates and surface charges and currents induced on the side $a > x > 0, y = 0$

Solution

The solution for E_z satisfying boundary condition $E_z\Big|_{x=0,a} = 0$ and $E_z\Big|_{y=0} = 0$ is

$$E_z = \sin\frac{\pi n x}{a}\sin k'y\; e^{ikz-i\omega t}$$

where $\mu\epsilon\omega^2 = \gamma^2 + k^2$ and $\gamma^2 = \frac{\pi^2 n^2}{a^2} + k'^2$. The transverse fields are

$$\vec{E}_T = \frac{ik}{\gamma^2}\vec{\nabla}_T E_z = \frac{ik}{\gamma^2}\Big(\frac{\pi n}{a}\cos\frac{\pi n x}{a}\sin k'y\;\hat{e}_1 + k'\sin\frac{\pi n x}{a}\cos k'y\;\hat{e}_2\Big)e^{ikz-i\omega t}$$

$$\vec{H}_T = \frac{i\omega\epsilon}{\gamma^2}\hat{e}_3\times\vec{\nabla}_T E_z = -\frac{\omega k}{\gamma^4}\Big(\frac{\pi n}{a}\cos\frac{\pi n x}{a}\sin k'y\;\hat{e}_1 + k'\sin\frac{\pi n x}{a}\sin k'y\;\hat{e}_2\Big)e^{ikz-i\omega t}$$

Boundary conditions on the $a > x > 0, y = 0$ surface are

$$\sigma = \Re\epsilon_0 E_2(x,0,z) = \frac{kk'}{\gamma^2}\sin\frac{\pi n x}{a}\sin(\omega t - kz)$$

$$\vec{K} = \Re\vec{H}\times\hat{e}_2 = -\frac{\omega k}{\gamma^4}\frac{\pi n}{a}\cos\frac{\pi n x}{a}\sin k'y\;\hat{e}_3$$

Problem 2.

A particle of charge q moves in a circle of radius a at a constant angular velocity ω. Assume that the circle lies in the x, y plane, centered at the origin and at time $t = 0$, the charge is at $(a, 0)$ on the positive x axis. For points on the z axis, find

(a) the Lienard-Wiechert potentials and

(b) the time-averaged electric field.

Solution

Part (a).

The Lienard-Wiechert potentials are given by

$$\phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{w}(t_r)| - \frac{1}{c}\vec{v}(t_r)\cdot(\vec{r}-\vec{w}(t_r))}$$
$$\vec{A}(\vec{r},t) = \frac{\vec{v}(t_r)}{c^2}\phi(\vec{r},t)$$

In our case $|\vec{r}-\vec{w}(t)| = \sqrt{z^2+a^2}$ and $\vec{v}(t)\cdot(\vec{r}-\vec{w}(t)) = 0$ for any t so we get

$$\phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0\sqrt{z^2+a^2}}$$
$$\vec{A}(\vec{r},t) = \frac{\vec{v}(t_r)}{c^2}\phi(\vec{r},t) \;=\; \frac{\mu_0 q}{4\pi\sqrt{z^2+a^2}}[-\sin(t-t_r)\hat{e}_1 + \cos(t-t_r)\hat{e}_2]$$

where $t_r = \frac{1}{c}\sqrt{z^2+a^2}$ is the retarded time.

Part (b).

By symmetry, the time-averaged electric field is collinear to the z axis so it is sufficient to find $E_z(z,0,0)$

$$\langle E_z(z,0,0)\rangle \;=\; -\langle\frac{\partial}{\partial z}\phi(z,0,0)\rangle - \langle\frac{\partial A_z}{\partial t}\rangle \;=\; -\frac{\partial}{\partial z}\phi(z,0,0) \;=\; \frac{qz}{4\pi\epsilon_0(z^2+a^2)^{3/2}}$$

so

$$\langle\vec{E}(z,0,0)\rangle \;=\; \frac{qz}{4\pi\epsilon_0(z^2+a^2)^{3/2}}\hat{e}_3$$

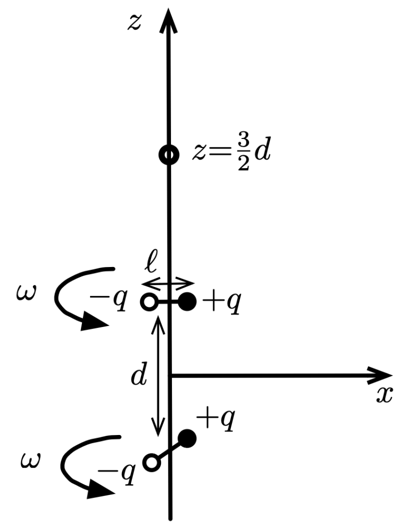
Problem 3.

Consider two identical rods of length l with charges $+q$ and $-q$ pasted on their ends. The centers of the rods are located on the z-axis which is perpendicular to the length of the rods (see below). The two rods are separated by a distance $d \gg l$, with the top and bottom rods located at heights $z = \pm\frac{d}{2}$ respectively. The rods rotate around the z axis with the same frequency ω but are out of phase, and at time $t=0$ the bottom rod has an azimuthal angle of ϕ_0 while the top rod has $\phi = 0$.

(a) In the limit $\frac{\omega}{c}d \gg 1$ and $\frac{\omega}{c}l \ll 1$, find (real) electric and magnetic fields at a height $z = \frac{3d}{2}$ on the z axis.

(b) Find the time-averaged total power radiated by this system.

Solution



Part (a).

The two rotating dipoles can be represented by

$$\vec{p}_1 = p_0(\hat{e}_1 + i\hat{e}_2)e^{-i\omega t}$$
$$\vec{p}_2 = p_0(\hat{e}_1 + i\hat{e}_2)e^{-i\omega t - i\phi_0}$$

The electric field is at the point $z = \frac{3d}{2}$ is the sum of the radiation fields of two dipoles separated from the observation point by d and $2d$ respectively. We get

$$\vec{E}_1 = -\frac{\mu_0}{4\pi}\omega^2\hat{n}\times(\hat{n}\times\vec{p}_1)\frac{e^{ikd}}{d}e^{-i\omega t} = \frac{\mu_0}{4\pi}\omega^2\vec{p}_1\frac{e^{ikd}}{d}e^{-i\omega t} = \frac{\mu_0}{4\pi}\omega^2 p_0(\hat{e}_1 + i\hat{e}_2)\frac{e^{ikd}}{d}e^{-i\omega t}$$
$$\vec{E}_2 = -\frac{\mu_0}{4\pi}\omega^2\hat{n}\times(\hat{n}\times\vec{p}_2)\frac{e^{2ikd}}{2d}e^{-i\omega t} = \frac{\mu_0}{4\pi}\omega^2\vec{p}_2\frac{e^{2ikd}}{2d}e^{-i\omega t} = \frac{\mu_0}{4\pi}\omega^2 p_0(\hat{e}_1 + i\hat{e}_2)\frac{e^{2ikd}}{2d}e^{-i\omega t - i\phi_0}$$

because $\hat{n} = \hat{e}_3$ is orthogonal to $\vec{p}_i$. So, because of the retardation, the phases of the two dipoles are $e^{-i\omega(t-\frac{d}{c})}$ and $e^{-i\omega(t-\frac{2d}{c}+\phi_0)}$.

Taking the real part, we obtain

$$\vec{E}_1 = \Re\frac{\mu_0}{4\pi}\omega^2 p_0(\hat{e}_1 + i\hat{e}_2)\frac{e^{ikd}}{d}e^{-i\omega t} = \frac{\mu_0\omega^2}{4\pi d}\Big[\hat{e}_1\cos\omega\Big(t - \frac{d}{c}\Big) + \hat{e}_2\sin\omega\Big(t - \frac{d}{c}\Big)\Big]$$
$$\vec{E}_2 = \Re\frac{\mu_0}{4\pi}\omega^2 p_0(\hat{e}_1 + i\hat{e}_2)\frac{e^{2ikd}}{2d}e^{-i\omega t - i\phi_0} = \frac{\mu_0\omega^2}{8\pi d}\Big[\hat{e}_1\cos\omega\Big(t - \frac{2d}{c} + \phi_0\Big) + \hat{e}_2\sin\omega\Big(t - \frac{2d}{c} + \phi_0\Big)\Big]$$

so the total electric field is

$$\vec{E}\Big(0,0,\frac{3d}{2}\Big) \;=\; \frac{\mu_0\omega^2}{4\pi d}\Big\{\hat{e}_1\Big[\cos\omega\Big(t-\frac{d}{c}\Big)+\frac{1}{2}\cos\omega\Big(t-\frac{2d}{c}+\phi_0\Big)\Big]+\hat{e}_2\Big[\sin\omega\Big(t-\frac{d}{c}\Big)+\frac{1}{2}\sin\omega\Big(t-\frac{2d}{c}+\phi_0\Big)\Big]\Big\}$$

and the magnetic field is

$$\vec{B}\Big(0,0,\frac{3d}{2}\Big) \;=\; \frac{\hat{e_3}}{c}\times\vec{E}\Big(0,0,\frac{3d}{2}\Big)$$

$$=\; \frac{\mu_0\omega^2}{4\pi dc}\Big\{\hat{e}_2\Big[\cos\omega\Big(t-\frac{d}{c}\Big)+\frac{1}{2}\cos\omega\Big(t-\frac{2d}{c}+\phi_0\Big)\Big]-\hat{e}_1\Big[\sin\omega\Big(t-\frac{d}{c}\Big)+\frac{1}{2}\sin\omega\Big(t-\frac{2d}{c}+\phi_0\Big)\Big]\Big\}$$

Part (b).

For the purpose of calculation of total radiated power, the two dipoles can be replaced by

$$\vec{p}=\vec{p}_1+\vec{p}_2 \;=\; p_0(\hat{e}_1+i\hat{e}_2)(1+e^{-i\phi_0})e^{-i\omega t} \;=\; 2p_0\cos\frac{\phi_0}{2}(\hat{e}_1+i\hat{e}_2)e^{-i\omega\left(t-\frac{\phi_0}{2}\right)}$$

Using the result for power radiated by rotating dipole from HW8, the time-averaged power radiated by this rotating dipole is

$$P \;=\; \frac{\mu_0\omega^4}{6\pi c}4p_0^2\cos^2\frac{\phi_0}{2} \;=\; \frac{\mu_0\omega^4}{3\pi c}p_0^2(1+\cos\phi_0)$$