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Quantum Mechanics of Relativistic Particles.

Lecture Notes

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1 Notations

I use the notations from the book by Peskin and Schoeder (except for arrows on Dirac lines). The units are $\hbar = c = 1$.

Also: Fourier transform - $(2\pi)^{-1}$ goes with $\int dk$:

$$f(x) = \int \frac{d^n k}{(2\pi)^n} e^{-ikx} f(k) \qquad f(k) = \int d^n x e^{ikx} f(x) \tag{1.1}$$

Dirac *delta*-function:

Definition:

$$\int dy\ \delta(x-y) f(y) = f(x) \tag{1.2}$$

In multi-dimensional space

$$\delta^{(n)}(x-y) \stackrel{\text{def}}{\equiv} \delta(x_1-y_1)\delta(x_2-y_2)...\delta(x_n-y_n) \tag{1.3}$$

Sometimes we will omit the upper label (n) for brevity (if there is no confusion about the dimension of the δ-function)).

Properties:

$$\delta(F(x) - F(y)) = \frac{1}{|F'(x)|}\delta(x-y) \tag{1.4}$$

θ-function:

$$\begin{aligned} \theta(x) &= 1 \quad x \geq 0 \\ \Theta(x) &= 0 \quad x < 0 \end{aligned} \tag{1.5}$$

Derivative:

$$\frac{d}{dx}\theta(x) = \delta(x) \tag{1.6}$$

Space-saving notation (inspired bu $\hbar = \frac{h}{2\pi}$):

$$\int \bar{d}^n p \ \equiv \ \int \frac{d^n p}{(2\pi)^n} \tag{1.7}$$

where n is the dimension ($n = 3$ for space and $n = 4$ for space-time).

Part I

2 Green functions in the non-relativistic scattering theory

2.1 Scattering in quantum mechanics

A scattering experiment correspond to the situation when we have imposed certain initial conditions in the remote past and we are intersted in the state of our affairs in the remote future. For example, a typical scattering problem studied in non-relativistic quantum mechanics is: given a wave packet which in the remote past represents a particle approaching the potential, one asks what the wave will look like in the remote future (see Fig. 1) Qualitatively we can view this process in terms of Huygens' principle. If the wave function

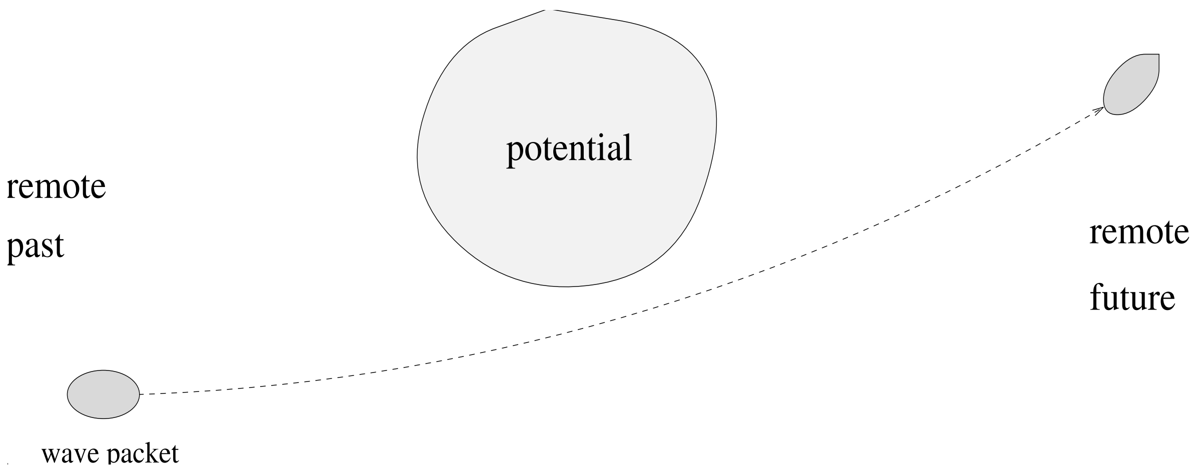


Figure 1. Typical scattering problem studied in the non-relativistic quantum mechanics

$\Psi(t_1, \vec{r}_1)$ is known at one particular time t_1, it may be found at any later time t_2 by considering at time t_1 each point of space $\vec{r}_1$ as a source of spherical waves which propagate outward from $\vec{r}_1$ (see Fig. **??**).

The strength of the wave amplitude arriving at point $\vec{r}_2$ at time t_2 from the point $\vec{r}_1$ will be proportional to the original wave amplitude $\Psi(t_1, \vec{r}_1)$. If we denote the constant of proportionality by $K(t_2, \vec{r}_2; t_1, \vec{r}_1)$, the total wave arriving at the point $\vec{r}_2$ at time t_2 will, by Huygens' principe, be

$$\Psi(t_2, \vec{r}_2) = \int d^3 r_1 K(t_2, \vec{r}_2; t_1, \vec{r}_1) \Psi(t_1, \vec{r}_1) \tag{2.1}$$

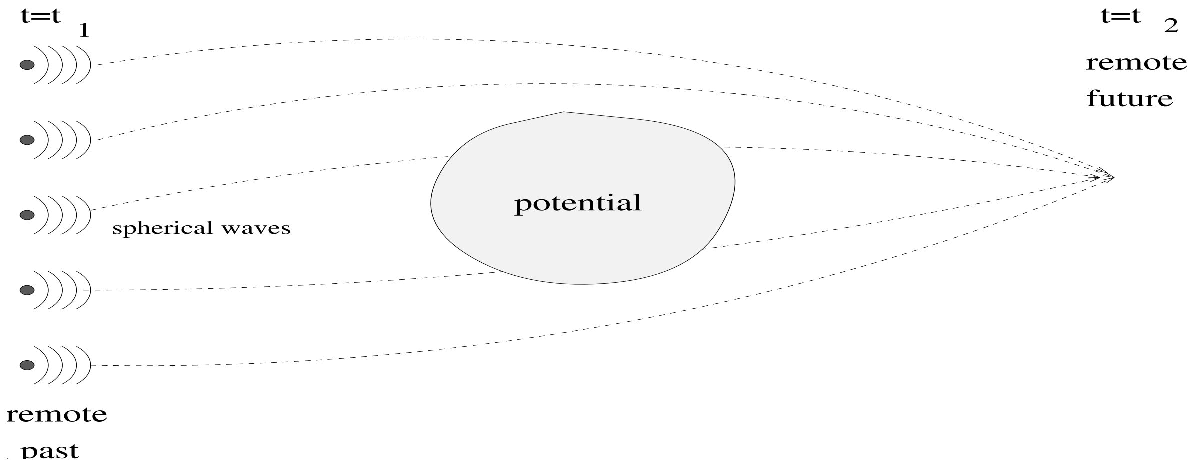


Figure 2. Qualitative description of a scattering in terms of Huygens' principle

The function $K(t_2, \vec{r}2; t_1, \vec{r}_1)$ is known as "the propagation amplitude". It describes the influence upon $\Psi(t_2, \vec{r}_2)$ of the magnitude of Ψ at the time t_1. Finding K is equivalent to the complete solution of the scattering problem; given the initial state, we can find the state of the system at arbitrary later time using eq.(2.1).

Quantitatively, we derive the formula (6.36) from the corresponding Schrödinger equation for the motion of the particle (wave packet) in the potential V:

$$i\frac{d}{dt}\Psi(t, \vec{r}) = H(t, \vec{r})\Psi \tag{2.2}$$

$$H(t, \vec{r}) = -\frac{\nabla^2}{2m} + V(t, \vec{r}) \tag{2.3}$$

(where, as usual, $\nabla_i \equiv \frac{d}{dx_i}$). We define of the propagation amplitude $K(t_2, \vec{r}_2; t_1, \vec{r}_1)$ as a solution of the Schrödinger equation

$$\left(i\frac{d}{dt_2} - H(t_2, \vec{r}_2)\right) K(t_2, \vec{r}_2; t_1, \vec{r}_1) = 0, \tag{2.4}$$

with the special initial condition:

$$K(t_1, \vec{r}_2; t_1, \vec{r}_1) = \delta(\vec{r}_2 - \vec{r}_1) \tag{2.5}$$

The physical meaning of the propagation amplitude is that it describes the fate of a particle that was created at the time t_1 in the point $\vec{r}_1$.

In the case of the stationary potential $V = V(\vec{r})$ it is easy to relate the propagation amplitude to the eigenfunctions of stationary Schrödinger equation. For the stationary case the time dependence of the solutions is trivial:

$$\Psi_n(t, \vec{r}) = e^{-iE_n t}\Psi_n(\vec{r}) \tag{2.6}$$

where the energies E_n are the eigenvalues of the stationary Schrödinger equation

$$H\Psi_n(\vec{r}) = E_n\Psi_n(\vec{r}) \tag{2.7}$$

Now, if we know a full set $\Psi_n(\vec{r})$ of eigenfunctions of the stationary Schrödinger equation (2.7) we can find the propagation amplitude using the following formula:

$$K(t_2,\vec{r}_2;t_1,\vec{r}_1) = \sum_n \Psi_n(t_2,\vec{r}_2)\Psi_n^*(t_1,\vec{r}_1) \tag{2.8}$$

It is easy to verify this formula. First, it trivially satisfies the Schrödinger equation (2.4) since each term in the sum in r.h.s. of eq. (2.8) does so. Second, the initial condition (2.5) is the completeness relation for the set of solutions of the stationary Schrödinger equation (2.7):

$$\sum_n \Psi_n(\vec{r}_2)\Psi_n^*(\vec{r}_1) = \delta(\vec{r}_2 - \vec{r}_1) \tag{2.9}$$

As an illustration, let us find the propagation function for the free particle. Let us consider one particle in the large box with side L [1]

The plane wave corresponding to the free non-relativistic particle moving in a box with size L (with periodical boundary conditions) has the form:

$$\Psi_{\vec{n}} = \frac{1}{\sqrt{L^3}} e^{-\frac{i}{2m}\vec{p}_{\vec{n}}^2 t + i\vec{p}_{\vec{n}}\vec{r}} \tag{2.10}$$

where $\vec{p}_{\vec{n}} = \frac{2\pi}{L}\vec{n}$, $\vec{n} = (n_1, n_2, n_3)$. These plane waves are normalized as usual eigenfunctions of the discrete spectrum:

$$\int d^3r \Psi_{\vec{m}}(t,\vec{r})\Psi_{\vec{n}}^*(t,\vec{r}) = \delta_{mn} \tag{2.11}$$

where δ_{mn} is equal to 1 if $m = n$ and 0 otherwise. The propagation amplitude for the free particle is given by the expression (2.8). Let us calculate it in the limit of very large box. When the box is very large one can replace the summation over the discrete spectrum of states (labeled by $\vec{n}$) by the integration over the continuous label $\vec{p}$

$$\sum_{\vec{n}} \to L^3 \int \frac{d^3p}{(2\pi)^3} \tag{2.12}$$

One obtains:

$$\begin{aligned} K_0(t_2,\vec{r}_2;t_1,\vec{r}_1) &\stackrel{\text{def}}{\equiv} \sum_{\vec{n}} \Psi_n(t_2,\vec{r}_2)\Psi_n^*(t_1,\vec{r}_1) \\ &\to L^3 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{L^3}} e^{-\frac{i}{2m}\vec{p}^2 t_2 + i\vec{p}\vec{r}_2} \frac{1}{\sqrt{L^3}} e^{\frac{i}{2m}\vec{p}^2 t_1 - i\vec{p}\vec{r}_1} = \\ &\int \frac{d^3p}{(2\pi)^3} e^{\frac{-i}{2m}\vec{p}^2 t_{21} + i\vec{p}\vec{r}_{21}} = \left(\frac{m}{2\pi i t}\right)^{3/2} e^{i\frac{r_{21}^2 m}{2t_{21}}} \end{aligned} \tag{2.13}$$

It is easy to check that (2.13) satisfies the Schrödinger equation (2.4) (at $V = 0$) with the initial condition

$$K_0(t_1,\vec{r}_2;t_1,\vec{r}_1) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{r}_{21}} = \delta(\vec{r}_2 - \vec{r}_1) \tag{2.14}$$

[1] Quite often we will consider the infinite space as a limited case of the large box with side $L \to \infty$ because it is convenient to start from a situation where we have a discrete spectrum rather than the continuouis one.

2.2 Non-relativistic particle in external field

Consider now the propagation of our non-relativistic particle in a certain external field described by the potential V. We will treat this potential as a small perturbation. Let us obtain the propagation amplitude as a series in powers of V.

In the trivial order in perturbation theory there is no any interaction with the external potential and the particle simply propagates freely (see Fig. 3) The free propagation

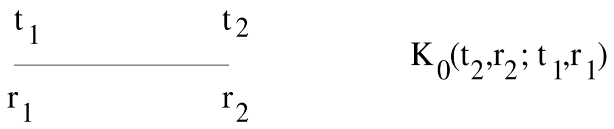


Figure 3.

amplitude satisfies the free Schrödinger equation:

$$\left(i\frac{d}{dt_2} - H_0\right) K_0(t_2, \vec{r}_2; t_1, \vec{r}_1) = 0 \tag{2.15}$$

In the second order we will take into account one interaction with the potential. The corresponding picture is shown in Fig. 4: the particle is created at time t_1 in the point

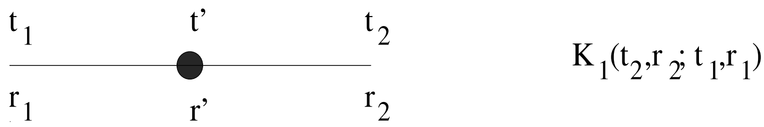


Figure 4.

$r = r_1$, then it propagates freely to the point r' where the interaction occurs at $t = t'$ then again the particle moves freely up to the end point r_2 where it is absorbed at the time t_2. This interaction can take place at any time $t_1 < t' < t_2$ in any point r'. Due to the superposition principle of quantum mechanics we must sum over all these possibilities so the propagation amplitude takes the form:

$$K_1(t_2, \vec{r}_2; t_1\vec{r}_1) = -i \int_{t_1}^{t2} dt' \int d^3r' K_0(t_2, \vec{r}_2; t', \vec{r}')V(t', \vec{r}')K_0(t', \vec{r}'; t_1, \vec{r}_1) \tag{2.16}$$

The coefficient $-i$ can be fixed from the perturbative form of the Schrödinger equation

$$i\frac{d}{dt}(K_0 + K_1 + K_2 + ...) = (H_0 + V)(K_0 + K_1 + K_2 + ...) \tag{2.17}$$

In the first order we get:

$$\left(i\frac{d}{dt_2} - H_0\right) K_1(t_2, \vec{r}_2; t_1, \vec{r}_1) = V(t_2, \vec{r}_2)K_0(t_2, \vec{r}_2; t_1, \vec{r}_1) \tag{2.18}$$

Actually, the eq. (2.16) is an educated guess which should be confirmed by check (2.18).

In the next order in perturbation theory we must take into account two interactions with the external potential (see Fig. 5). The corresponding contribution to the propagating

Figure 5. Non-relativistric propagation amplitude in the second order

amplitude has the form:

$$K_2(t_2,\vec{r}_2;t_1,\vec{r}_1) = \tag{2.19}$$
$$\int_{t_1}^{t_2} dt'' \int_{t_1}^{t''} dt' \int d^3r' d^3r" K_0(t_2,\vec{r}_2;t",\vec{r}")[-i]V(t",\vec{r}")K_0(t",\vec{r}";t,'\vec{r}')[-i]V(t',\vec{r}')K_0(t',\vec{r}';t_1,\vec{r}_1)$$

It is easy to see that it satisfies the Schrödinger equation

$$\left(i\frac{d}{dt_2} - H_0\right) K_2(t_2,\vec{r}_2;t_1,\vec{r}_1) = V(t_2,\vec{r}_2)K_1(t_2,\vec{r}_2;t_1,\vec{r}_1) \tag{2.20}$$

which justifies our educated guess (2.19) for K_2.

Continuing this procedure, one obtains the total propagation amplitude in external potential as an infinite sum over the potential insertions:

$$K(t_2,\vec{r}_2;t_1,\vec{r}_1) = \sum_{n=0}^{\infty} K_n(t_2,\vec{r}_2;t_1,\vec{r}_1) \tag{2.21}$$

where K_n satisfies the equation

$$(i\frac{d}{dt_2} - H_0)K_n(t_2,\vec{r}_2;t_1,\vec{r}_1) = V(t_2,\vec{r}_2)K_{n-1}(t_2,\vec{r}_2;t_1,\vec{r}_1) \tag{2.22}$$

By construction, the sum (2.21) satisfies the Schrödinger equation (2.17) with the initial condition (2.5)

$$K(t_1,\vec{r}_2;t_1,\vec{r}_1) = K_0(t_1,\vec{r}_2;t_1,\vec{r}_1) = \delta(\vec{r}_2 - \vec{r}_1) \tag{2.23}$$

(all other K_n vanish at $t_2 = t_1$).

It is convenient to introduce the so-called Green function which is defined in the following way:

$$G(t_2,\vec{r}_2;t_1,\vec{r}_1) \stackrel{\text{def}}{\equiv} \theta(t_2 - t_1)K(t_2,\vec{r}_2;t_1,\vec{r}_1) \tag{2.24}$$

This Green function satisfies the inhomogenuous Schrödinger equation:

$$\left(i\frac{d}{dt_2} - H(\vec{r}_2,t_2)\right)G(t_2,\vec{r}_2;t_1\vec{r}_1) = \tag{2.25}$$
$$= \; K(t_2,\vec{r}_2;t_1,\vec{r}_1)i\tfrac{d}{dt_2}\theta(t_2-t_1) = i\delta(t_2-t_1)K_0(t_1,\vec{r}_2;t_1,\vec{r}_1) = i\delta(t_2-t_1)\delta(\vec{r}_2-\vec{r}_1)$$

2.3 Feynman diagrams for Green functions

In short, the Green function can be represented by the same diagrams as the propagation function (see Figs. 3,4, 5). The analytical expressions corresponding to these diagrams are the same up to one distinction: in terms of propagation functions, we must put the limits

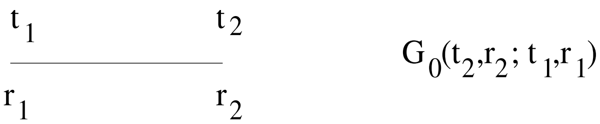


Figure 6.

of integration over times of interaction by hand; in terms of Green functions the cutoffs due to causality are automatic so we just put time limits from $-\infty$ to ∞.

For completeness, let us redraw these diagrams. The first diagram is:

The free Green function corresponds to the motion without interaction with the external potential. A single interaction is desribed by the diagram similar to Fig. (4):

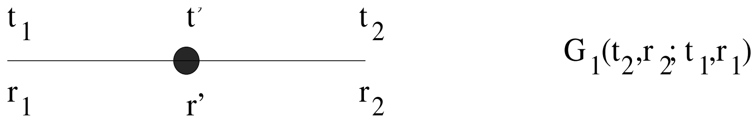


Figure 7.

The corresponding analytical expression has the form (cf. eq. (2.16)):

$$G_1(t_2,\vec{r}_2;t_1,\vec{r}_1) = -i\int dt'\int d^3r' G_0(t_2,\vec{r}_2;t',\vec{r}')V(t',\vec{r}')G_0(t',\vec{r}';t_1,\vec{r}_1) \tag{2.26}$$

In the second order the diagram is the same as Fig. (5) and the explicit form of G_2 (cf.

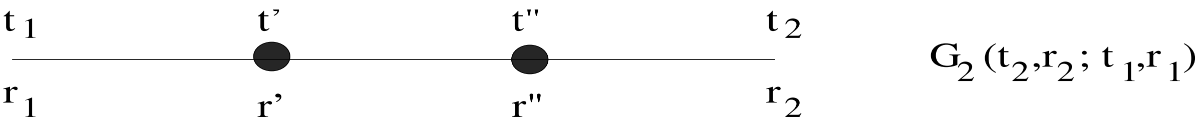


Figure 8. Non-relativistric Green function in the second order

eq.(2.19)) is:

$$G_2(t_2,\vec{r}_2;t_1,\vec{r}_1) = \tag{2.27}$$
$$\int dt'dt" \int d^3r'd^3r" G_0(t_2,\vec{r}_2;t",\vec{r}")[-i]V(t",\vec{r}")G_0(t",\vec{r}";t',\vec{r}')[-i]V(t',\vec{r}')G_0(t',\vec{r}';t_1,\vec{r}_1)$$

The $n-$th term of this expansion can be represented by a similar Feynman diagram with $n-1$ insertions shown in Fig. 9. The corresponding analytical expresssion contains

Figure 9. A Green function in arbitrary order of perturbation theory

$(n-1)$ integrations.

The formal sum of this series of diagrams is the total Green Function $G(t_2, \vec{r}_2; t_1, \vec{r}_1)$ defined in eq. (2.24).

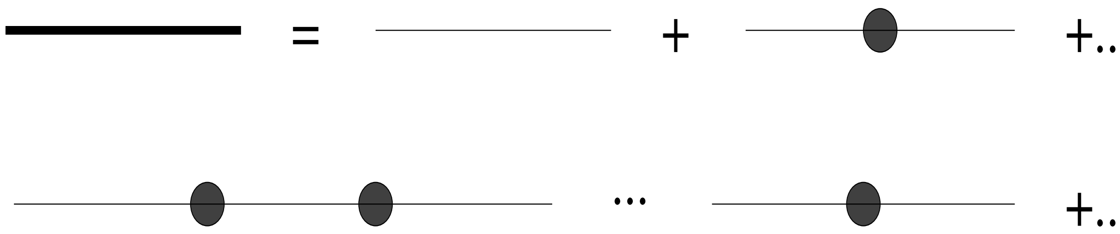


Figure 10. Total Green function as infinite sum of Feynman diargams

To prove it, note that the formal sum

$$G(t_2, \vec{r}_2; t_1, \vec{r}_1) = \sum_{n=0}^{\infty} G_n(t_2, \vec{r}_2; t, \vec{r}') \tag{2.28}$$

satisfies the following integral equation illustrated in Fig. 11:

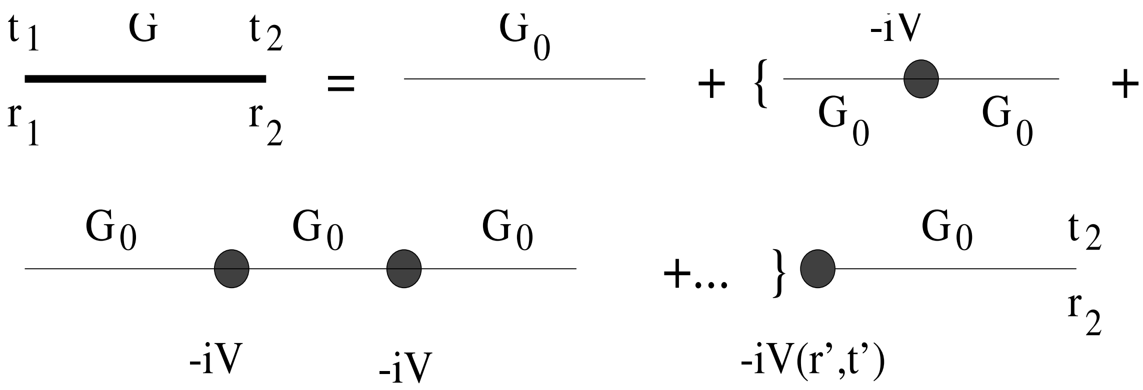


Figure 11. Integral equation for exact Green function

$$G(t_2, \vec{r}_2; t_1, \vec{r}_1) = G_0(t_2, \vec{r}_2; t_1, \vec{r}_1) - i \int dt' \int d^3r' G_0(t_2, \vec{r}_2; t', \vec{r}') V(t', \vec{r}') G(t', \vec{r}'; t_1, \vec{r}_1) \tag{2.29}$$

If we apply the operator $i\frac{\partial}{\partial t} - H_0$ to both sides of this equation we get

$$\begin{aligned}
&(i\frac{\partial}{\partial t_2} - H_0) G(t_2, \vec{r}_2; t_1, \vec{r}_1) = \\
&i\delta(\vec{r}_2 - \vec{r}_1)\delta(t_2 - t_1) + \int dt' \int d^3r' \delta(\vec{r}_2 - \vec{r}')\delta(t_2 - t') V(t', \vec{r}') G(t', \vec{r}'; t_1, \vec{r}_1) = \\
&i\delta(\vec{r}_2 - \vec{r}_1)\delta(t_2 - t_1) + V(\vec{r}_2, t_2) G(\vec{r}_2, t_2; \vec{r}_1, t_1)
\end{aligned} \tag{2.30}$$

which is equivalent to eq. (2.25)

Part II

2.4 Non-relativistric Green functions in the momentum representation

To warm up, let us show that the free propagator in momentum space has the familiar form $(\frac{p^2}{2m} - p_0)^{-1}$. (Warning: a propagator is another word for a Green function and *not* for the propagation amplitude). The differential equation for the free propagator is eq. (2.25) with $H \to H_0$:

$$\left(i\frac{d}{dt_2} + \frac{\nabla^2}{2m} \right) G(t_2, \vec{r}_2; t_1, \vec{r}_1) = i\delta(t_2 - t_1)\delta(\vec{r}_2 - \vec{r}_1) \tag{2.31}$$

By Fourier transformation we get

$$G(t_2, \vec{r}_2; t_1\vec{r}_1) = \int \frac{d^4p}{16\pi^4 i} \frac{1}{\frac{\vec{p}^2}{2m} - p_0} e^{-ip_0 t_{21} + i\vec{p}\vec{r}_{21}} \tag{2.32}$$

While performing the integration over p_0 we face a problem: the denominator has a pole at $p_0 = \frac{\vec{p}^2}{2m}$. In order to regularize this divergence it is convenient to add a small imaginary number $-i\epsilon$ to the expression in the denominator which shifts the pole to the lower half of the p_0 plane (see Fig. 12):

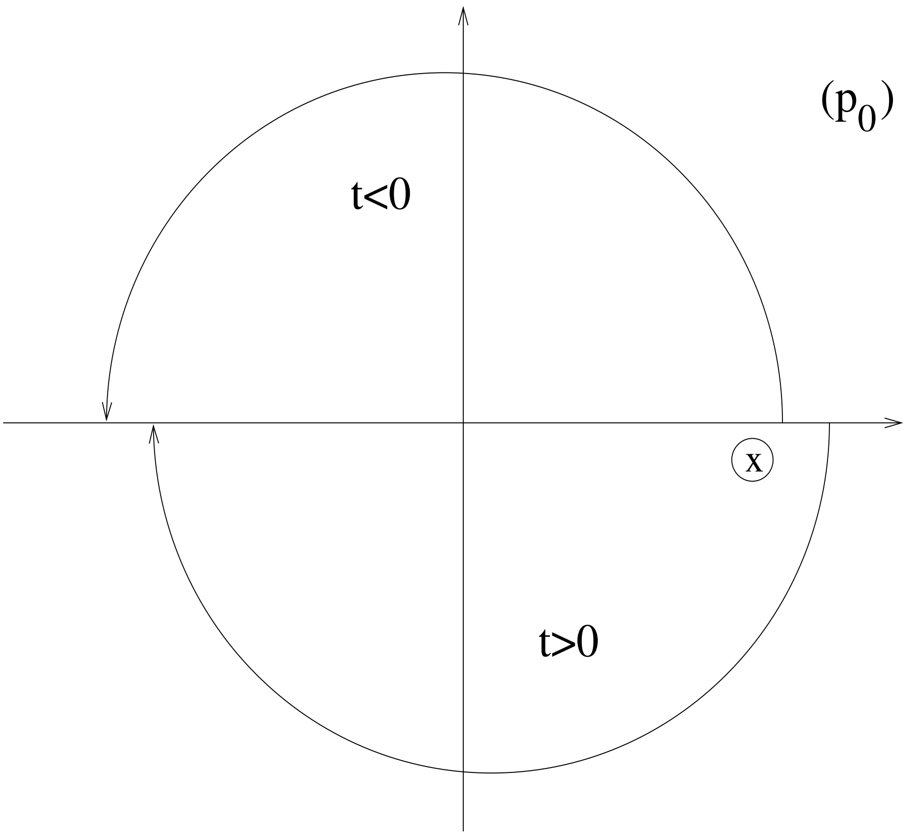


Figure 12. Calculation of the integral over p_0

After that we get:

$$\lim|_{\epsilon\to 0}\int dp_0\frac{1}{\frac{\vec p^2}{2m}-p_0-i\epsilon}e^{-ip_0t_{21}}=$$
$$\theta(t_{21})\cdot 2\pi i\mathrm{Res}|_{p_0=\vec p^2/2m}=2\pi i e^{-i\frac{\vec p^2}{2m}t_{21}}\theta(t_{21}) \tag{2.33}$$

The remaining integral over $\vec p$ is trivial so one obtains:

$$\lim|_{\epsilon\to 0}\int\frac{d^4p}{16\pi^4 i}\frac{1}{\frac{\vec p^2}{2m}-p_0-i\epsilon}e^{-ip_0t_{21}+i\vec p\vec r_{21}}=$$
$$\theta(t)\int\frac{d^3p}{8\pi^3}e^{-i\frac{\vec p^2}{2m}t+i\vec p\vec r_{21}}=\left(\frac{2m}{it\pi}\right)^{3/2}e^{i\frac{r_{21}^2 m}{2t_{21}}} \tag{2.34}$$

which coincides with $G_0=\theta(t)K_0$, see eq. (2.13). It is worth noting that the regulator $i\epsilon$ in the denominator enforces the condition

$$G_0(t_2,\vec r_2;t_1,\vec r_1)=0 \qquad \text{at} \;\; t_2<t_1 \tag{2.35}$$

Should we choose the regulator $i\epsilon$ rather than $-i\epsilon$ we would get the function proportional to $\theta(-t)$ instead. Thus, the free propagator in the momentum representation is

$$G_0(p)=\frac{1}{\frac{p^2}{2m}-p_0-i\epsilon} \tag{2.36}$$

Next we consider the propagator in the first order in perturbation theory (2.26). It is convenient to introduce the notation $V(p)$ for the Fourier transform of the potential:

$$V(\vec r,t)=\int\frac{d^4q}{16\pi^4}e^{-iq_0t+i\vec q\vec r}V(q) \tag{2.37}$$

Using this notation, one can rewrite the first-order contribution (2.26) in the form:

$$\int d^3r'dt'\left(\int\frac{d^4p_2}{16\pi^4 i}G_0(p_2)e^{-ip_{20}(t_2-t')+i\vec p_2(\vec r_2-\vec r')}\right)$$
$$\int\frac{d^4q}{16\pi^4}e^{-iq_0t'+i\vec q\vec r'}(-iV(q))\left(\int\frac{d^4p_1}{16\pi^4 i}G_0(p_1)e^{-ip_{10}(t'-t_1)+i\vec p_1(\vec r'-\vec r_1)}\right) \tag{2.38}$$

The integration over r' and t' gives the corresponding δ-functions:

$$\int d^3r'e^{i(\vec q+\vec p_1-\vec p_2)\vec r'}\int dt'e^{i(p_{20}-p_{10}-q_0)t'}=$$
$$(2\pi)^3\delta^{(3)}(\vec p_1-\vec p_2)(2\pi)\delta(p_{20}-p_{10})=(2\pi^4)\delta^{(4)}(q+p_1-p_2) \tag{2.39}$$

Performing the integration over q we obtain:

$$G_1(t_2,\vec r_2;t_1,\vec r_1)=\int\frac{d^4p_1d^4p_2}{(2\pi)^8 i}e^{-ip_{20}t_2+i\vec p_2\vec r_2}e^{ip_{10}t_1-i\vec p_1\vec r_1}G_0(p_2)[-V(p_2-p_1)]G_0(p_1) \tag{2.40}$$

This expression can be represented by the following diagram where lines denote the free propagators in the momentum space (2.36) and the vertex is the Fourier transform of the potential (2.37).

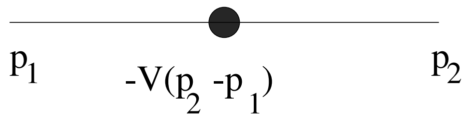


Figure 13.

Let us now introduce the exact Green function in the momentum representation :

$$G(\vec{r}_2, t_2; \vec{r}_1, t_1) = \int \frac{d^4p_1 d^4p_2}{(2\pi)^8 i} e^{-ip_{20}t_2 + i\vec{p}_2\vec{r}_2} e^{ip_{10}t_1 - i\vec{p}_1\vec{r}_1} G(p_1, p_2) \tag{2.41}$$

The two first terms of the perturbative expansion of Green function are

$$G(p_1, p_2) = (2\pi)^4 \delta^{(4)}(p_1 - p_2) G_0(p_1) + G_0(p_2)[-V(p_2 - p_1)]G_0(p_1) + ... \tag{2.42}$$

The equation (2.42) in terms of diagrams is shown in Fig. 14.

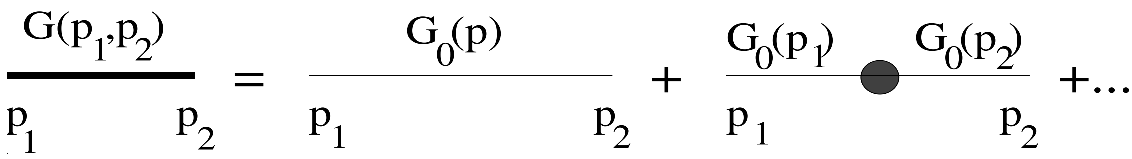
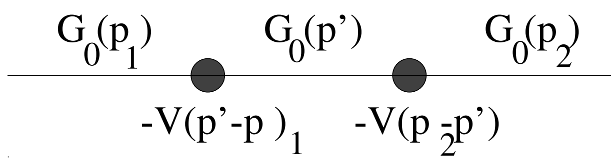


Figure 14. Green function in the momentum representation

In the next order the expression for the Green function takes the form

$$G_2(p_1, p_2) = G_0(p_2) \int \frac{d^4p'}{(2\pi)^4} V(p_2 - p') G_0(p') V(p' - p_1) G_0(p_1) \tag{2.43}$$

which corresponds to the second-order diagram



It worth noting that $G_1(p_1, p_2)$ (the second term in eq. (2.42)) corresponds to the first Born approximation in the usual perturbation series in quantum mechanics. Similarly, eq. (2.43) corresponds to the second Born approximation (and the integral over p' stands for the sum over intermediate states in the usual formula for the second Born term).

Finally, the perturbative series for the Green function in the momentum representation has the form:

$$\begin{aligned} G(p_1, p_2) &= (2\pi)^4 \delta(p_1 - p_2) G_0(p_1) + \\ & G_0(p_2)[-V((p_2 - p_1)]G_0(p_1) + G_0(p_2) \int \frac{d^4p'}{(2\pi)^4)} V(p_2 - p') G_0(p') V(p' - p_1) G_0(p_1) \ + \ ... \end{aligned} \tag{2.44}$$

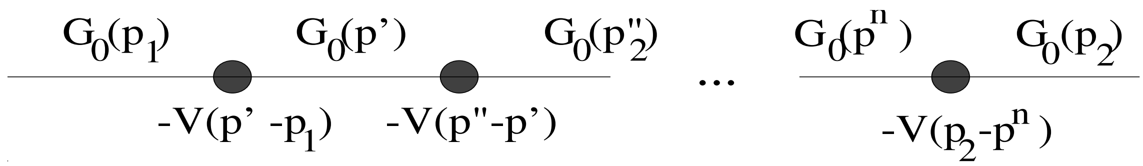


Figure 15. Feynman rules for non-relativistic Green functions

Looking at this series we can formulate the Feynman rules for the n-th order diagram for the non-relativistic Green function

1. We write down the factor $G_0(p_n)$ for each line
2. Each vertex comes with the factor $-V(p_n, p_{n-1})$
3. There is an integration $\int \frac{d^4 p_n}{(2\pi)^4}$ corresponding to each internal line

Finally, we must sum up all the diagrams to obtain the total Green function $G(p_2, p_1) = \sum_0^\infty G_n(p_2, p_1)$.

2.5 Scattering Matrix and Green functions

Let us calculate the probability of the transition from a certain initial state $\Psi_i(\vec{r}, t)$ to the final state described by the wave function $\Psi_f(\vec{r}, t)$. If the initial state at the time t_1 was $\Psi_i(\vec{r}, t)$ then at the moment of time t_2 the wave function of the system is

$$\begin{aligned}\Psi(\vec{r}_2, t_2) &= \int d^3 r_1 K(\vec{r}_2, t_2; \vec{r}_1, t_1)\Psi_i(\vec{r}_1, t_1) \\ &= \int d^3 r_1 G(\vec{r}_2, t_2; \vec{r}_1, t_1)\Psi_i(\vec{r}_1, t_1)\end{aligned} \tag{2.45}$$

Using our definition of the propagation functon $\mathcal{K}$ it is easy to check that the r.h.s. of eq. (2.45) satisfies Schrödinger equation with the proper initial condition $\Psi(\vec{r}, t_1) = \Psi_i(\vec{r}, t_1)$. The amplitude to discover this particle (described by wavefunction (2.45)) at the time t_2 in the state $\Psi_f(\vec{r}, t)$ is

$$\int d^3 r_2 \Psi_f^*(\vec{r}_2, t_2)\Psi(\vec{r}_2, t_2) \tag{2.46}$$

so the amplitude of the transition $i \to f$ takes the form

$$\int d^3 r_2 \Psi_f^*(\vec{r}_2, t_2)\Psi(\vec{r}_2, t_2) = \int d^3 r_1 d^3 r_2 \Psi_f^*(\vec{r}_2, t_2) G(\vec{r}_2, t_2; \vec{r}_1, t_1)\Psi_i(\vec{r}_1, t_1) \tag{2.47}$$

The scattering experiment corresponds to the situation when $t_1 \to -\infty$ (remote past) and $t_2 \to \infty$ (remote future) so the amplitude of the $i \to f$ scattering has the form

$$S_{fi} = \lim_{t_2\to\infty} \lim_{t_1\to-\infty} \int d^3 r_1 d^3 r_2 \Psi_f^*(\vec{r}_2, t_2) G(\vec{r}_2, t_2; \vec{r}_1, t_1)\Psi_i(\vec{r}_1, t_1) \tag{2.48}$$

This transition amplitude is called “matrix element of scattering matrix” (or S-matrix for short) and it is the main subject of the calculations in scattering theory.

Consider a typical scattering process like Coulomb scattering. In the far past, long time before the scattering when $t_i \to -\infty$ the wave function was a plane wave with the momentum $\vec{p}_1$ and we would like to calculate the probability to find (long after the scattering) a free particle moving with the momentum $\vec{p}_2$ [2].

So,

$$\Psi_i(\vec{r}, t) = e^{-i\frac{\vec{p}_1^2}{2m}t + i\vec{p}_1\vec{r}} \tag{2.49}$$

$$\Psi_f(\vec{r}, t) = e^{-i\frac{\vec{p}_2^2}{2m}t + i\vec{p}_2\vec{r}} \tag{2.50}$$

Then the general expression for the matrix element of the S-matrix (2.48) reduces to:

$$\begin{aligned} S(\vec{p}_2, \vec{p}_1) = \int d^3r_2 d^3r_1 e^{i\frac{\vec{p}_2^2}{2m}t_2 - i\vec{p}_2\vec{r}_2} e^{-i\frac{\vec{p}_1^2}{2m}t_1 + i\vec{p}_1\vec{r}_1} \\ \int \frac{d^4p_1' d^4p_2'}{(2\pi)^8 i} G(p_1', p_2') e^{-ip_{20}'t_2 + i\vec{p'}_2\vec{r}_2} e^{ip_{10}'t_1 - i\vec{p'}_1\vec{r}_1} \end{aligned} \tag{2.51}$$

where we have used the formula (2.41). The integration over r_1 and r_2 gives

$$(2\pi)^3\delta(\vec{p}_2 - \vec{p'}_2)(2\pi)^3\delta(\vec{p}_1 - \vec{p'}_1) \tag{2.52}$$

so we obtain

$$S(\vec{p}_2, \vec{p}_1) = \frac{1}{4\pi^2 i}\int dp_{10}dp_{20} e^{it_2(\frac{\vec{p}_2^2}{2m} - p_{20})} e^{-it_1(\frac{\vec{p}_1^2}{2m} - p_{10})} G(p_{10}, \vec{p}_1; p_{20}, \vec{p}_2) \tag{2.53}$$

(we have wiped $'$ from the notation for the variables of integration). Let us recall now the expansion (2.42) of the Green function $G(p_1, p_2)$ in powers of the interaction potential

$$\begin{aligned} G(p_1, p_2) = (2\pi)^4\delta(p_1 - p_2)G_0(p_1) + G_0(p_2)[-V((p_2 - p_1)]G_0(p_1) + \\ + G_0(p_2)\int \frac{d^4p'}{(2\pi^4)} V(p_2 - p')G_0(p')V(p' - p_1)G_0(p_1) + \ldots \end{aligned} \tag{2.54}$$

Next, we substitute this expansion in the expression (2.53) for the S-matrix ellement. First term of the expansion yields:

$$\begin{aligned} S_0(\vec{p}_2, \vec{p}_1) &= -(2\pi)^2 i\delta(\vec{p}_1 - \vec{p}_2)\int dp_{10} e^{i(t_2 - t_1)(\frac{\vec{p}_1^2}{2m} - p_{10})} G_0(\vec{p}, p_{10}) \\ &= -(2\pi)^2 i\delta(\vec{p}_1 - \vec{p}_2)\int dp_{10} e^{i(t_2 - t_1)(\frac{\vec{p}_1^2}{2m} - p_{10})} \frac{1}{\frac{\vec{p}_1^2}{2m} - p_{10} - i\epsilon} = (2\pi)^3\delta(\vec{p}_1 - \vec{p}_2) \end{aligned} \tag{2.55}$$

which corresponds to the free propagation without scattering. Here again we have added the contour integral over the lower semicircle in the complex p_{10} plane (which can be done since $t_{21} > 0$) so the resulting integral over the closed contour is determined by the residue and is equal to $2\pi i$.

---

[2] Our idea about scattering is that in the distant past a particle moves freely, then it scatters by the potential which has a finite size and then moves freely again. To be self-consistent we have to think that even in the case of potential which is independent of time it switches off adiabatically slowly in distant past so that free wave is a true solution when $t \to -\infty$ and the potential is not yet switched on. It is not evident that such approach is without fault but more elaborate treatments show that everything is OK at this point.

All other terms in the expansion of the Green function $G(p_1, p_2)$ (see (2.54)) depend on both p_1 and p_2 so we may write down the Green function in the following form:

$$G(p_1, p_2) = (2\pi)^4 \delta(p_1 - p_2) G_0(p_1) + G_0(p_2)(2\pi)^4 T(p_1, p_2) G_0(p_1) \tag{2.56}$$

where the "transition matrix" $T(p_1, p_2)$ includes all vertices and internal lines (which also means the integration over all internal momenta). The expansion of the transition matrix in powers of V starts from the Born term:

$$(2\pi)^4 T(p_1, p_2) = -V(p_2 - p_1) + \int \frac{d^4 p'}{(2\pi^4)} V(p_2 - p') G_0(p') V(p' - p_1) + ... \tag{2.57}$$

Let us substitute this expression for $G(p_1, p_2)$ in the integrand in eq. (2.53). We have already calculated the first trivial term without scattering - see eq. (2.55). As to the second term, it has the form

$$\frac{1}{4\pi^2 i} \int dp_{10} dp_{20} e^{it_2(\frac{\vec{p}_2^2}{2m} - p_{20})} e^{-it_1(\frac{\vec{p}_1^2}{2m} - p_{10})} \frac{(2\pi)^4 T(p_1, p_2)}{(\frac{\vec{p}_1^2}{2m} - p_{10} - i\epsilon)(\frac{\vec{p}_2^2}{2m} - p_{20} - i\epsilon)} \tag{2.58}$$

We must calculate now the integrals over p_{10} and p_{20}. Let us start from the p_{10} integral. As usual, we add the contour integral over the lower semicircle in the complex p_{10} plane (it is easy to check that $e^{ip_{20}t}$ decreases in this direction) so the integral over p_{20} is determined by the singularities of the integrand. These are the pole $\frac{1}{\frac{\vec{p}_2^2}{2m} - p_{20} - i\epsilon}$ plus any other possible singularities of the function $T(p_1, p_2)$ which depend on the explicit form of the potential V. However, it is easy to see, however, that the contributions from any singularity other than the pole $\frac{1}{\frac{\vec{p}_2^2}{2m} - p_{20} - i\epsilon}$ vanishes at large t_2 due to the rapid oscillations of the exponent $e^{it_2(...)}$ so the integral over p_{20} gives

$$2\pi i T(p_1, p_2)\Big|_{p_{20} = \frac{\vec{p}_2^2}{2m}} \tag{2.59}$$

Repeating the same procedure for the integration over p_{10} we finally obtain:

$$S(\vec{p}_2, \vec{p}_1) = (2\pi)^3 \delta(\vec{p}_1 - \vec{p}_2) + (2\pi)^4 i T(\vec{p}_1, \vec{p}_2) \tag{2.60}$$

where

$$T(\vec{p}_1, \vec{p}_2) = T(p_1, p_2)\Big|_{p_{20} = \frac{\vec{p}_2^2}{2m},\ p_{10} = \frac{\vec{p}_1^2}{2m}} \tag{2.61}$$

The first term in eq. (2.60) corresponds to the free propagation and the second (T) determines the amplitude of the scattering. For example, in the first Born approximation

$$T(\vec{p}_1, \vec{p}_2) = -\frac{1}{(2\pi)^4} V(\frac{\vec{p}_2^2}{2m} - \frac{\vec{p}_1^2}{2m}, \vec{p}_2 - \vec{p}_1) \tag{2.62}$$

Let us formulate the final set of Feynman rules for the scattering amplitude $T(\vec{p}_1, \vec{p}_2)$:

1. Write down the diagrams

2. Write down the corresponding Green function using the rules formulated in Sect. A.1. (the factor $G_0(p_n)$ for each line, the factor $-V(p_n, p_{n-1})$ for each vertex and the integration $\int \frac{d^4 p_n}{(2\pi)^4}$ for each internal line). The answer will have the form:

$$G(p_1, p_2) = (2\pi)^4 \delta(p_1 - p_2) \frac{1}{\frac{\vec{p}_1^2}{2m} - p_{10} - i\epsilon} + \frac{(2\pi)^4 T(p_1, p_2)}{(\frac{\vec{p}_1^2}{2m} - p_{10} - i\epsilon)(\frac{\vec{p}_2^2}{2m} - p_{20} - i\epsilon)} \qquad (2.63)$$

3. The scattering amplitude $T(\vec{p}_1, \vec{p}_2)$ is $\frac{1}{(2\pi)^4}$ times the numerator in the second term in the above equation taken at $p_{10} = \frac{\vec{p}_1^2}{2m}$, $p_{20} = \frac{\vec{p}_2^2}{2m}$

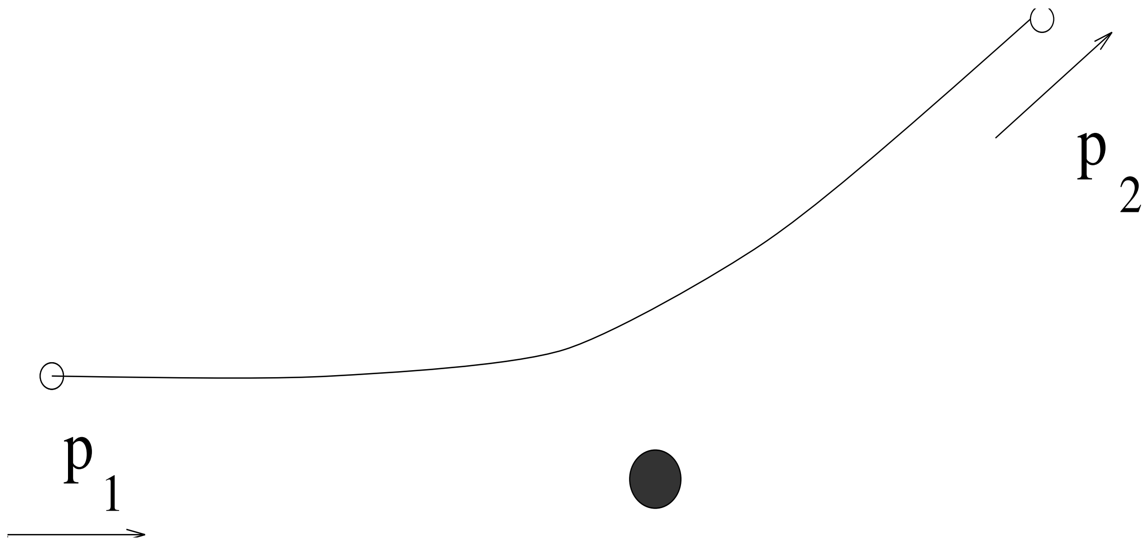
Part III

2.6 Coulomb scattering

I would like to calculate *cross section* for scattering of an electron by Coulomb potential in the first Born approximation. Suppose we have the electon scattereing from the repulsive Coulomb potential

$$V(\vec{r}) = \frac{Ze^2}{4\pi|\vec{r}|} \qquad (2.64)$$

corresponding to a certain large nuclei with atomic number Z [3]. Let me remind you the logic of our approach.



In the far past we have the plane wave corresponding to the incoming electron: with momentum $\vec{p}_1$

$$\tilde{\Psi}_{p_1}(\vec{r}, t) = \frac{1}{L^{3/2}} e^{-i\frac{\vec{p}_1^2}{2m}t + i\vec{p}_1\vec{r}} \qquad (2.65)$$

[3] In this chapter we will ignore the complexifications due to the fact that electron is a Dirac particle with the *spin* - we shall take it into account later when we will study the Dirac equation. As a result, we will get the Rutherford formulla for the cross section of the unpolarized scattering (I hope that in the end of this course you'll have no problem to understand what this means!)

and we are interested in the probability to observe this state at the remote future as a free electron with momentum $\vec{p}_2$ described by the plane wave [4]

$$\tilde{\Psi}_{p_2}(\vec{r},t) = \frac{1}{L^{3/2}} e^{-i\frac{\vec{p}_2^2}{2m}t + i\vec{p}_2\vec{r}} \tag{2.66}$$

Here it is convenient to normalize the wavefunction by the condition that we have only one particle in the large box with the side L:

$$\int_{-L/2}^{L/2} d^3r |\tilde{\Psi}_i(\vec{r},t)|^2 = 1 \tag{2.67}$$

Note that the normalization in the previous section was different: we had the plane waves normalized by the condition $\int d^3r \Psi^\dagger_{\vec{p}}(\vec{r},t)\Psi_{\vec{p}'}(\vec{r},t) = \delta(\vec{p}-\vec{p}')$ corresponding to the continuous spectrum. Because of that, the S-matrix will also differ from the previous Section by factor $1/L^3$ (see below). To clarify the notations, in this Section we will label wave functions, matrix elements, etc corresponding to the plane waves normalized according to eq. (2.67) by a tilde $\tilde{}$.

Time evolution of the electron during the scattering is given by the solution of the Schrödinger equation

$$\tilde{\Psi}^{(p_1)}(\vec{r}_2, t_2) = \int d^3r_1 G(\vec{r}_2, t_2; \vec{r}_1, t_1)\tilde{\Psi}_{p_1}(\vec{r}_1, t_1) \tag{2.68}$$

(see eq.(2.45)) which coincides with the plane wave with momentum p_1 at $t_1 \to -\infty$. After the scattering ($\equiv$ at the time $t_2 \to \infty$) this solution can be written down as a superposition of the outgoing pane waves

$$\tilde{\Psi}^{(p_1)}(\vec{r}, t_2) = L^3 \int \frac{d^3p_2}{(2\pi)^3} \tilde{S}(\vec{p}_1, \vec{p}_2)\tilde{\Psi}_{p_2}(\vec{r}, t_2) \tag{2.69}$$

with the coefficients being matrix elements of the S-matrix defined in Eq. (2.48):

$$\tilde{S}(\vec{p}_1, \vec{p}_2) = \int d^3r \tilde{\Psi}^*_{p_2}(\vec{r}, t_2)\tilde{\Psi}^{(p_1)}(\vec{r}, t_2) \tag{2.70}$$

The factor L^3 in Eq. (2.69) is due to the completeness condition (2.71) below. As we mentioned in the footnote, the Fourier transformation in Eq. (2.41) in the finite-volume space contains summation over $\vec{p}_n = \frac{2\pi}{L}\vec{n}$ rather than the integration over continuous $\vec{p}$ (see Sect. 2.1.A). The completeness condition in a finite volume reads as

$$\delta(\vec{r}-\vec{r}') = \sum_{\vec{n}} \tilde{\Psi}^*_{\vec{n}}(\vec{r}', t)\tilde{\Psi}_{\vec{n}}(\vec{r}, t)), \qquad \tilde{\Psi}_{\vec{p}_n}(\vec{r}, t) = \frac{1}{L^{3/2}} e^{-i\frac{\vec{p}_n^2}{2m}t + i\vec{p}_n\vec{r}}$$

but at large L we can use the completeness condition in the form

$$\delta(\vec{r}-\vec{r}') = L^3 \int \frac{d^3p}{(2\pi)^3} \tilde{\Psi}^*_p(\vec{r}', t)\tilde{\Psi}_p(\vec{r}, t) \tag{2.71}$$

[4] Strictly speaking, the fact that we consider this problem in a box will lead to the quantization of the electron momentum: $\vec{p}_{\vec{n}} = \frac{2\pi}{L}\vec{n}$ but we assume that the box is very large so the electon momentum changes almost continuously.

where the plane waves are normalized according to eq. (2.67).

Thus, the probability to find this electron moving with some momentum p_2 in the remote future is determined by the matrix element of the S-matrix which has the form

$$\tilde{S}(\vec{p_1}, \vec{p_2}) = \lim_{t_2 \to \infty} \lim_{t_1 \to -\infty} \int d^3 r_1 d^3 r_2 \tilde{\Psi}^*_{p_2}(\vec{r_2}, t_2) G(\vec{r_2}, t_2; \vec{r_1}, t_1) \tilde{\Psi}_{p_1}(\vec{r_1}, t_1) \tag{2.72}$$

where $\tilde{\Psi}_{p_2}$ is the wavefunction corresponding to the outgoing plane wave with momentum p_2 and $G(\vec{r_2}, t_2; \vec{r_1}, t_1)$ is the Green function for the Coulomb potential. Note that $\tilde{S}(\vec{p_1}, \vec{p_2}) = \frac{1}{L^3} S(\vec{p_1}, \vec{p_2})$.

Since the electric charge is very small ($\frac{e^2}{4\pi} = \frac{1}{137}$) it is enough to consider the first Born approximation. The Green function in the first Born approximation is

$$G_1(p_1, p_2) = (2\pi)^4 \delta(p_1 - p_2) G_0(p_1) - G_0(p_2) V(p_2 - p_1) G_0(p_1) \tag{2.73}$$

where the Fourier transform of the Coulomb potential has the form

$$V(p_2 - p_1) = 2\pi \delta(p_{10} - p_{20}) \frac{Ze^2}{|\vec{p_2} - \vec{p_1}|^2} \tag{2.74}$$

so the transition amplitude $T(\vec{p_2}, \vec{p_1})$ reduces to (see equations (2.56) and (2.62)):

$$\tilde{T}(\vec{p_1}, \vec{p_2}) = \frac{1}{L^3} T(\vec{p_1}, \vec{p_2}) = -\frac{1}{(2\pi)^4 L^3} 2\pi \delta(E_2 - E_1) \frac{Ze^2}{|\vec{p_2} - \vec{p_1}|^2}. \tag{2.75}$$

where $E_1 = \frac{\vec{p_1}^2}{2m}$ is the energy of the non-relativistic electron with momentum p_1 (and $E_2 = \frac{\vec{p_2}^2}{2m}$). Transition amplitude squared is equal to the probability of transition from the state with initial momentum $\vec{p_1}$ into the state with final momentum $\vec{p_2}$ anywhere in the space during the infinitely large time $\mathcal{T} = t_2 - t_1$. This probability depends of course on the flux of incoming particles. It is much more convenient to have some characteristic of scattering which is independent of the initial flux and which characterises intensity of scattering per unit of time.The probability is determined by the square of the matrix element of the S-matrix so the transition rate per unit time interval is [5]

$$P_{fi} = \frac{|(2\pi)^4 \tilde{T}(\vec{p_1}, \vec{p_2})|^2}{\mathcal{T}}. \tag{2.76}$$

The cross section is obtained from the scattering rate by dividing by the initial flux and multiplying by the number of final states. Initial flux is equal to

$$j = |\vec{p_1}|/(mL^3) \tag{2.77}$$

in accordance with normalization of initial state wave functions in Eq. 2.67. Indeed, total number of particles is normalized to be equal to 1 in volume L^3 and they move forward with momentum $\vec{p_1}$ or velocity $\vec{v_1} = \vec{p_1}/m$. Let us consider the cube (with side L) with our

[5] Working in terms of T-matrix rather than S-matrix may seem inconvenient, but only for the non-relativistic scattering: in the relativistic situatuion S-matrix contains the factor $(2\pi)^4$ is explicitly and to extract it from the definition is very natural.

particle inside it. The particle moves with velocity v_1 so after the time $t = \frac{L}{v}$ elapses, the particle should be somewhere inside the adjacent cube (with sides L). Hence, the number of the particles which cross the side of the cube with area L^2 in time $t = \frac{L}{v}$ is exactly 1 so the flux is:

$$\text{Flux} = \frac{\text{Number of particles}}{\text{Time} \otimes \text{Area}} = \frac{1}{tL^2} = \frac{v_1}{L^3} \tag{2.78}$$

The number of final states in a volume L^3 in momentum interval $d^3\vec{p}_2$ is equal to $L^3 d^3p_2/(2\pi)^3$ (see Lecture 1). Cross section is thus equal to

$$\begin{aligned}\sigma &= \frac{1}{j} L^3 \int \frac{d^3p_2}{(2\pi)^3} P_{fi} \\ &= \frac{mL^3}{|\vec{p}_1|} L^3 \int \frac{d^3p_2}{(2\pi)^3} \frac{|(2\pi)^4 \tilde{T}(\vec{p}_1, \vec{p}_2)|^2}{\mathcal{T}}\end{aligned} \tag{2.79}$$

Note that cross section has the dimension of area and admits a nice physical interpretation – it is equal to the effective target area to be hit by the particle in order to be scattered into the interval of finite states just described.

The r.h.s. of Eq. (2.79) contains the square of a δ-function which is ill-defined so one has to provide it with some sense. To this end one has simply to recall the integral representation for the δ-function

$$\delta(E_2 - E_1) = \frac{1}{2\pi} \int dt \exp[it(E_2 - E_1)] \tag{2.80}$$

Next, it is almost evident that the square of δ-function in Eq. (2.79) can be interpreted as follows:

$$\begin{aligned}\{\delta(E_2 - E_1)\}^2 &= \frac{1}{2\pi} \int dt\ e^{t(E_2 - E_1)} \frac{1}{2\pi} \int dt'\ e^{it'(E_2 - E_1)} \\ &= \frac{1}{2\pi} \int dt\ \delta(E_2 - E_1) e^{it(E_2 - E_1)} = \frac{1}{2\pi} \int dt\ \delta(E_2 - E_1) \\ &= \frac{\mathcal{T}}{2\pi} \delta(E_2 - E_1).\end{aligned} \tag{2.81}$$

After substitution of the above expression to Eq. (2.79) one obtains

$$\begin{aligned}\sigma &= \int \frac{1}{(L^3)^2} (2\pi)^2 \frac{\mathcal{T}}{2\pi} \delta(E_2 - E_1) \Big(\frac{Ze^2}{|\vec{p}_2 - p_1|^2} \Big)^2 \frac{1}{\mathcal{T}} \frac{mL^3}{|\vec{p}_1|} \frac{L^3 d^3\vec{p}_2}{(2\pi)^3} \\ &= \frac{mZ^2e^4}{4\pi^2} \int d^3p_2 \frac{1}{|\vec{p}_1|} \frac{1}{|\vec{p}_2 - \vec{p}_1|^4} \delta(E_2 - E_1).\end{aligned} \tag{2.82}$$

In the general case of a time-independent potential $V = V(\vec{r})$ the cross section has the form:

$$\sigma = \frac{1}{4\pi^2 v_1} \int d^3p_2\ \delta(E_2 - E_1) |V(\vec{p}_2 - \vec{p}_1)|^2 \tag{2.83}$$

where $V(\vec{q})$ isa a three-dimensional Fourirer transform of the potential:

$$\begin{aligned}V(q) &= \int d^4x\ V(x) e^{iqx} = 2\pi\delta(E_2 - E_1) V(\vec{q}) \\ V(\vec{q}) &= \int d^3r\ V(\vec{r}) e^{-i\vec{q}\vec{r}}\end{aligned} \tag{2.84}$$

Next step is to take into account that $d^3p_2 = p_2^2 dp_2 d\Omega$ where the solid angle element is defined by the relation $d\Omega = d\phi \sin(\theta) d\theta$ (it is assumed here that the z-axis is defined by the direction of the momentum $\vec{p}_1$). Then δ-function of energies leads to the equality of absolute values of initial and finite momenta $|\vec{p}_1| = |\vec{p}_2| = p = mv$ and $(\vec{p}_1 - \vec{p}_2)^2 = 4p^2 \sin^2(\theta/2)$. Using these substitutions one easily converts Eq. (2.82) to familiar Rutherford formula for the differential cross section of the Coulomb scattering:

$$\sigma = \int d\Omega \, \frac{d\sigma}{d\Omega}, \tag{2.85}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{8\pi m v^2}\right)^2 \frac{1}{\sin^4(\theta/2)} = \frac{Z^2\alpha^2}{4m^2v^4} \frac{1}{\sin^4(\theta/2)},$$

where $\alpha = e^2/4\pi \approx 1/137$ is called the fine structure constant (in the usual units $\alpha = \frac{e^2}{4\pi\hbar c}$). It is a universal dimensionless constant which characterizes strength of electromagnetic interaction and it enters all QED results.

Remarkable fact about the Rutherford formula is that for Coulomb scattering in Classical Mechanics and Born approximation in Quantum Mechanics lead to the same result! In all other situations results of these theories coincide only in some limiting cases.

Homework assignment 1.

Nuclear forces are described by Yukawa potential:

$$V(r) = V_0 \frac{e^{-\alpha|\vec{r}|}}{|\vec{r}|} \tag{2.86}$$

where V_0 and α are real constants, with α positive. This potential is repulsive or attractive depending on the sign of V_0. [6]

Problem 1. Find the differential cross section $\frac{d\sigma}{d\Omega}$ (analog of the Rutherford formula) for the scattering from Yukawa potential.

Problem 2. Find σ - the *total cross section* (see eqs. (2.79),(2.82)) for the scattering from Yukawa potential.

Problem 3. What is the total cross section for the scattering from Coulomb potential?

2.7 Unitarity of S-matrix

I have demonstrated that the probability amplitude is determined by matrix element (f, i) of a certain matrix S which is called the scattering matrix. This matrix is unitary and this feature of the scattering matrix is of particular importance. Unitarity means that

$$\sum_n S_{ln} S^\dagger_{nk} = \delta_{kl}, \tag{2.87}$$

or in matrix notation

$$SS^\dagger = 1. \tag{2.88}$$

[6] To explain the origin of this potential, Yukawa was led to predict the existence of π-meson, which was indeed discovered sometime later.

It is very easy to prove unitarity. Consider for simplicity the case of discrete spectrum in the initial (and final) state. (A good example is plane waves in the finite box and it is easy to generalize our proof to the case of continuous spectrum of plane waves in the infinite space). Let $\tilde{\Psi}^{(k)}(\vec{r},t)$ and $\tilde{\Psi}^{(l)}(\vec{r},t)$ be two solutions of Schrödinger equation which coincide with two plane waves $\tilde{\Psi}_k(\vec{r},t)$ and $\tilde{\Psi}_l(\vec{r},t)$ at the time $t=t_1$ ($\equiv$ in the remote past) :

$$\tilde{\Psi}^{(k)}(\vec{r},t) = \int d^3r' G(\vec{r},t;\vec{r}',t_1)\tilde{\Psi}_k(\vec{r}',t_1) \tag{2.89}$$

and similarly for l. Actually, the indices k,l, etc. are the vector ones: $\vec{k}=(k_1,k_2,k_3)$ but we will omit the vector sign here for brevity. Due to the hermiticity of the Hamiltonian H the scalar product of the two solutions conserves in time. Indeed,

$$i\frac{d}{dt}\int d^3r\tilde{\Psi}^{\dagger(k)}(\vec{r},t)\tilde{\Psi}^{(l)}(\vec{r},t) = \int d^3r\tilde{\Psi}^{\dagger(k)}(\vec{r},t)(-H+H)(\tilde{\Psi}^{(l)})(\vec{r},t) = 0 \tag{2.90}$$

so

$$\int d^3r\tilde{\Psi}^{\dagger(k)}(\vec{r},t)\tilde{\Psi}^{(l)}(\vec{r},t) = \int d^3r\tilde{\Psi}^{\dagger}_k(\vec{r},t_1)\tilde{\Psi}_l(\vec{r},t_1) = \delta_{kl} \tag{2.91}$$

Now, we can expand the wavefunction $\tilde{\Psi}^{(k)}(\vec{r},t)$ at the time $t=t_2$ ($\equiv$ remote future) over the complete set of plane waves:

$$\tilde{\Psi}^{(k)}(\vec{r},t_2) = \sum_n \tilde{S}_{kn}\tilde{\Psi}_n(\vec{r},t_2) \tag{2.92}$$

This formula can be obtained from our definition of the matrix elements of the S-matrix (2.48) which can be rewritten in this case as follows:

$$\tilde{S}_{kn} = \int d^3r\tilde{\Psi}^{\dagger}_n(\vec{r},t_2)\tilde{\Psi}^{(k)}(\vec{r},t_2) \tag{2.93}$$

and from the condition of the completeness of the eigenfunctions $\tilde{\Psi}_n(\vec{r},t_2)$ which has the form

$$\delta(\vec{r}-\vec{r}') = \sum_n .\tilde{\Psi}^{\dagger}_n(\vec{r},t)\tilde{\Psi}_n(\vec{r},t) \tag{2.94}$$

(for arbitrary t). Substituting now the expansion (2.92) in the equation (2.91) we finally obtain

$$\sum_n \tilde{S}_{ln}\tilde{S}^{\dagger}_{nk} = \delta_{kl} \tag{2.95}$$

where we have used again the property of the orthogonality of the eigenfunctions

$$\int d^3r\tilde{\Psi}^{\dagger}_m(\vec{r},t_2)\tilde{\Psi}_n(\vec{r},t_2) = \delta_{mn}$$

.

Requirement of unitarity is one of the most important in quantum physics and has a transparent physical meaning. Let us consider diagonal in k,k term in Eq. (2.87)

$$\tilde{S}^{\dagger}_{kk}\tilde{S}_{kk} + \sum_{n\neq k}\tilde{S}^{\dagger}_{kn}\tilde{S}_{nk} = 1. \tag{2.96}$$

This simply means that there is no leakage of probability during the transition. First term in Eq. (2.96) is the probability that the state k survives the collision and second term is equal to the total probability of transition from the state k to any other state. Obviously sum of these terms should be equal to one from physical considerations. Nondiagonal term in Eq. (2.87) should be equal to zero as we just have seen that this vanishing of nondiagonal terms simply reflects orthogonality of different initial state wave functions.

Unitarity of the S-matrix is valid for any physical system. We will later consider field theories where the analogues of the Schrödinger wave function satisfy nonlinear equations but unitarity still survives. Quite often in the theories with strong interactions the unitarity ($\equiv$ the optical theorem) is the only way to calculate the total cross section.

Let us now recover the optical theorem in quantum mechanics. For the case of continuous spectrum the unitarity condition (2.95) takes the form:

$$\int \frac{d^3p}{(2\pi)^3} S(\vec{k},\vec{p})S^\dagger(\vec{p},\vec{q}) = (2\pi)^3\delta(\vec{k}-\vec{q}) \tag{2.97}$$

Let us substitute here the S-matrix in the form (2.60)

$$S(\vec{k},\vec{p}) = (2\pi)^3\delta(\vec{k}-\vec{p}) + (2\pi)^4 iT(\vec{k},\vec{p}) \tag{2.98}$$

(and similarly for $S(\vec{p},\vec{q})$. It is easy to see that the term proportional to the product of two δ functions gives exactly the r.h.s. of eq. (2.97) so the sum of three remaining terms must be zero. We obtain then :

$$\frac{1}{i}(T(\vec{k},\vec{q}) - T^*(\vec{k},\vec{q})) = (2\pi)^4 \int \frac{d^3p}{(2\pi)^3} T(\vec{k},\vec{p})T^*(\vec{p},\vec{q}) \tag{2.99}$$

At $k = q$ (for the case of forward scattering) this reduces to:

$$\Im T(k,k) = \frac{1}{2}(2\pi)^4 \int \frac{d^3p}{(2\pi)^3} |T(\vec{k},\vec{p})|^2 \tag{2.100}$$

In the case of scattering from the stationary potential the transition matrix $T(\vec{p}_1,\vec{p}_2)$ is related to the usual non-relativistic scattering amplitude $f(\vec{p}_1,\vec{p}_2)$ in the following way:

$$T(\vec{p}_1,\vec{p}_2) = \frac{1}{(2\pi)^3 m} 2\pi\delta(E_1 - E_2) f(\vec{p}_1,\vec{p}_2) \tag{2.101}$$

Using the "mnemonic rule" $2\pi\delta(0) = \mathcal{T}$ we can rewrite the Eq. (2.100) as

$$\Im f(k,k) = \frac{m}{2\mathcal{T}}(2\pi)^7 \int \frac{d^3p}{(2\pi)^3} |T(\vec{k},\vec{p})|^2 \tag{2.102}$$

The (total) cross section is obtained by substitution of this equation to Eq. (2.79)

$$\sigma_{\rm tot} = \frac{mL^3}{|\vec{k}|} L^3 \int \frac{d^3p_2}{(2\pi)^3} \frac{|(2\pi)^4\tilde{T}(\vec{k},\vec{p})|^2}{\mathcal{T}} = \frac{m}{|\vec{k}|\mathcal{T}}(2\pi)^8 \int \frac{d^3p_2}{(2\pi)^3} |T(\vec{k},\vec{p})|^2 \tag{2.103}$$

where we used $\tilde{T}(\vec{p}_1, \vec{p}_2) = \frac{1}{L^3} T(\vec{p}_1, \vec{p}_2)$, see Eq. (2.75). Now, from the comparison of Eq. (2.101) and Eq. (2.103) we see that the optical theorem (2.100) reduces to a standard textbook form

$$\Im f(\vec{k}, \vec{q}) \;=\; \frac{|\vec{k}|}{4\pi}\sigma_{\rm tot} \tag{2.104}$$

It is worth noting that in the relativistic theory the optical theorem (which looks very similar to eq. (2.104) is the main tool for calculations of the total cross sections of various processes such as deep inelastic electron-proton scattering.

Part IV

3 Relativistic quantum mechanics

3.1 Quantum mechanics of a scalar meson

In the framework of classical relativistic mechanics the π-meson field is described by a Klein-Gordon (KG) equation

$$(\Box + m^2)\phi(x) = 0, \tag{3.1}$$

reflecting the relation between energy and momentum for the massive relativistic particle

$$E = \sqrt{m^2 + \vec{p}^2} \tag{3.2}$$

We will consider the case of neutral π-meson so the corresponding KG field $\phi(x)$ is real. Unfortunately, it is impossible to observe classical π-meson field since the mass of the π-mesonm=140 MeV is very large. (The force due to the pion field decreases as e^{-mr} so it is negligible beyond $r = 1$fm$\simeq 10^{-13}cm$). If we lived in another world with $m_\pi \sim 10^{-7}$eV, the force due to classical π-mesonfield should be observable at distances $\sim$ 1m so it would be discovered long time ago and the Klein-Gordon equation would be a part of the course on general physics.

The KG equation describe the propagation of a free π-meson wave in the empty space. The general solution of this equation is the superposition of the plane waves with the four-momenta p lying "on the mass shell" ($\Leftrightarrow p^2 = m^2$). Indeed, the corresponding equation for the Fourier transform

$$\phi(p) = \int d^4x \; e^{ipx}\phi(x) \tag{3.3}$$

($kx \equiv k^\mu x_\mu$) has the form

$$(p^2 - m^2)\phi(p) = 0 \tag{3.4}$$

which means that

$$\phi(p_0, \vec{p}) = \delta(p^2 - m^2)F(p_0, \vec{p}) \tag{3.5}$$

where F is some function of p_0 and $\vec{p}$. The equation $p^2 = m^2$ has two solutions: $p_0 = E_p$ and $p_0 = -E_p$ where $E_p = \sqrt{\vec{p}^2 + m^2}$. Keeping in mind that our original function $\phi(x)$ must be real we obtain now that the general form of the function $\phi(p)$ is

$$\phi(p_0, \vec{p}) = 2\pi\delta(p_0 - E_p)\Phi(\vec{p}) + 2\pi\delta(p_0 + E_p)\Phi^*(-\vec{p}) \tag{3.6}$$

where Φ is an arbitrary function of $\vec{p}$. In the coordinate space the general solution (3.6) is

$$\phi(x) = \phi_+(x) + \phi_-(x) \tag{3.7}$$

where

$$\begin{aligned} \phi_+(x) &= \int \frac{d^3p}{(2\pi)^3} e^{-iE_pt+i\vec{p}\vec{r}}\Phi(\vec{p}) \\ \phi_-(x) &= \int \frac{d^3p}{(2\pi)^3} e^{iE_pt+i\vec{p}\vec{r}}\Phi^*(-\vec{p}) \end{aligned} \tag{3.8}$$

The functions ϕ_+ and ϕ_- are called positive- and negative-frequency parts of the field (since in the first case the frequency $p_0 = E_p$ is positive while in the second case it is negative: $p_0 = -E_p$. By construction, each of them satisfies the KG wave equation (3.1).

Let us try to develop quantum mechanical description for this free π-meson field. Although we will obtain no new results in comparison to classical description [7] the quantum formalism for the free wave is an important ingredient for description of the interactions of particles (where there *are* new results in comparison to the classical physics!).

In the quantum mechanical approach a particle is described by the wave function $\Psi(x)$ and the square of the wave function is the density of the probability distribution. We assume that the KG field is made from the π-mesons described by quantum mechanics and try to construct the wave function Ψ for these mesons. The hope is to find some quantity which has the probabilistic interpretation like

$$\int d^3r|\Psi(\vec{r,t})|^2 = 1 \tag{3.9}$$

If this quantity is conserved it may serve as a probability to observe a π-meson (anywhere in the space). On the other hand, the desired wave function $\Psi(x)$ must satisfy the equation (3.1) which reflects the relation (3.2) between energy and momentum of a relativistic particle.

Let us recall how we get the conservation of the probability in the non-relativistic quantum mechanics (cf. Sect.1.3). Let us write down the Schrödinger equation and the complex conjugate of it

$$i\frac{d}{dt}\Psi = H\Psi \qquad\qquad H = H_0 = \frac{\vec{p}^2}{2m} = -\frac{\vec{\nabla}^2}{2m} \tag{3.10}$$

$$-i\frac{d}{dt}\Psi^* = H\Psi^* \tag{3.11}$$

(as usual, $\nabla_i \equiv \frac{d}{dx_i}$). Now it is easy to see that

$$i\frac{d}{dt}\Psi^*(\vec{r},t)\Psi(\vec{r},t) = \frac{1}{2m}[\Psi(\vec{r},t)\vec{\nabla}^2\Psi^*(\vec{r},t) - \Psi^*(\vec{r},t)\vec{\nabla}^2\Psi(\vec{r},t)] \tag{3.12}$$

[7] One should expect this since the role of quantum effects enter the game when the feedback of the classical detector on the π-meson field, i.e. the interaction, is nonzero.

or, in other words,

$$\frac{d}{dt}\Psi^*(\vec{r},t)\Psi(\vec{r},t) = -\frac{i}{2m}\nabla_j[\Psi(\vec{r},t)\nabla_j\Psi^*(\vec{r},t) - \Psi^*(\vec{r},t)\nabla_j\Psi(\vec{r},t)] \tag{3.13}$$

and the total integral of the r.h.s. of this equation over the whole space vanishes after integration by parts. Thus, we have two properties of the quantity $\Psi^*\Psi$:

$$\begin{aligned} &1)\ \Psi^*(\vec{r},t)\Psi(\vec{r},t) > 0 \\ &2)\ \int d^3r\Psi^*(\vec{r},t)\Psi(\vec{r},t) = const \end{aligned} \tag{3.14}$$

which give us an opportunity to interpret $|\Psi(\vec{r},t)|^2$ as a density of the probability (to find a particle in a point $\vec{r}$ at the time t). Indeed, the first equation means that the density of the probability is positive while the second ensures that the total probability to observe the particle anywhere in the space is 1.

Now we will try to construct smth which has a meaning of probability to observe a π-meson. Instead of the Schrödinger eqn we have now the wave equation (3.1) so we need a quantity which satisfies the continuity equation of the eq. (3.13) type. The (educated) guess is to take the positive-frequency part $\phi_+(x)$ of the classical meson field. [8] Let us try to repeat the derivation of the eq. (3.14) for this case.

We write down two equations - for ϕ_+ and ϕ_+^* :

$$\begin{aligned} &\tfrac{d^2}{dt^2}\phi_+(\vec{r},t) + m^2\phi_+(\vec{r},t) - \vec{\nabla}^2\phi_+(\vec{r},t) = 0 \\ &\tfrac{d^2}{dt^2}\phi_+^*(\vec{r},t) + m^2\phi_+^*(\vec{r},t) - \vec{\nabla}^2\phi_+^*(\vec{r},t) = 0 \end{aligned} \tag{3.15}$$

Muiltiplying the first of these equations by ϕ_+^* and subtracting the second multiplied by ϕ, we obtain:

$$\frac{d}{dt}[\phi_+^*(t,\vec{r})\frac{d}{dt}\phi_+(t,\vec{r}) - \phi_+(t,\vec{r})\frac{d}{dt}\phi_+^*(t,\vec{r})] = \nabla_j[\phi_+^*(t,\vec{r})\nabla_j\phi_+(t,\vec{r}) - \phi_+(t,\vec{r})\nabla_j\phi_+^*(t,\vec{r})] \tag{3.16}$$

so

$$\begin{aligned} &\frac{d}{dt}\int d^3r[\phi_+^*(t,\vec{r})\frac{d}{dt}\phi_+(t,\vec{r}) - \phi_+(t,\vec{r})\frac{d}{dt}\phi_+^*(t,\vec{r})] = \\ &= \int d^3r\nabla_j[\phi_+^*(t,\vec{r})\nabla_j\phi_+(t,\vec{r}) - \phi_+(t,\vec{r})\nabla_j\phi_+^*(t,\vec{r})] = 0 \end{aligned} \tag{3.17}$$

since the integral in the r.h.s. over the whole space vanishes after integration by parts. We see that the integral

$$\int d^3r[\phi_+^*(t,\vec{r})\frac{d}{dt}\phi_+(t,\vec{r}) - \phi_+(t,\vec{r})\frac{d}{dt}\phi_+^*(t,\vec{r})] \tag{3.18}$$

is conserved so we may try to interpret the quantity in square brackets as a probability density. We define [9]:

$$\rho(t,\vec{r}) = i\phi_+^*(t,\vec{r})\frac{d}{dt}\phi_+(t,\vec{r}) - i\phi_+(t,\vec{r})\frac{d}{dt}\phi_+^*(t,\vec{r}) \stackrel{\text{def}}{\equiv} \phi_+^* i \overset{\leftrightarrow}{\frac{d}{dt}} \phi_+ \tag{3.19}$$

[8] Alternatively, one can take a negative-frequency part; however, one cannot take the meson field ϕ itself for the reasons discussed below (see Eq. (3.19).

[9] Not that for the KG field itself this quantity is zero because the KG field is real — that is the reason why we take the positive-frequenbcy part of it $\phi_+(x)$ rather than the field itself.

As we demonstrated, the integral of this quantity over the whole space is conserved. To go ahead with probabilistic interpretation, we must show that it is also positive (cf. eq. (3.14)). We have

$$\int d^3r\rho(t,\vec{r}) =$$
$$\int d^3r \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3} i\phi(\vec{p})\phi^*(\vec{p'})[e^{iE_pt-i\vec{p}\vec{r}}(-iE'_p)e^{-iE'_pt+i\vec{p}'\vec{r}} - e^{-iE'_pt+i\vec{p}'\vec{r}}(-iE_p)e^{iE_pt-i\vec{p}\vec{r}}$$
$$= \int \frac{d^3p}{(2\pi)^3} 2E_p\phi(\vec{p})\phi^*(\vec{p}) \tag{3.20}$$

which is surely positive.

The last remaining issue is whether we can ascribe a meaning of the *local* probability density for the function $\rho(x)$. In general, the function (3.19) can have arbitrary sign, but for the stationary case described by wave function

$$\phi_+(x) = e^{-i\omega t}\phi(\vec{r}) \tag{3.21}$$

we have

$$\rho(t,\vec{r}) = 2\omega|\phi(\vec{r})|^2 \tag{3.22}$$

which means that for stationary state the function (3.19) has the meaning of the probability density as in the non-relativistic quantum mechanics. In general, we cannot define the probability distribution for the system of π-mesons which merely reflects the fact that in the relativistic quantum mechanics the number of the particles is not conserved.

So, one π-meson in a box with side L is described by the wave function

$$\phi_+(x) = \frac{1}{\sqrt{2E_pL^3}}e^{-iE_pt+i\vec{p}\vec{r}} \tag{3.23}$$

It is easy to see that

$$\frac{1}{2E_pL^3}\int_{-L}^{L} d^3r e^{iE_pt-i\vec{p}\vec{r}} i \overset{\leftrightarrow}{\frac{d}{dt}} e^{-iE_pt+i\vec{p}\vec{r}} = 1 \tag{3.24}$$

$$\frac{1}{2E_pL^3}\int_{-L}^{L} d^3r e^{iE_pt-i\vec{p}\vec{r}} i \overset{\leftrightarrow}{\frac{d}{dt}} e^{-i|\vec{p'}|t+i\vec{p'}\vec{r}} = 0, \qquad p \neq p' \tag{3.25}$$

(Strictly speaking, the momenta in a box are quantized : $\vec{p}_{\vec{n}} = \frac{2\pi}{L}\vec{n}$ and the integral (3.25) vanishes for $\vec{n} \neq \vec{n}'$). In practice it is more convenient to use wave functions without the $\frac{1}{L^{3/2}}$ factor.

$$\phi_{\vec{p}}(x) = \frac{1}{\sqrt{2E_p}}e^{-iE_pt+i\vec{p}\vec{r}} \tag{3.26}$$

These wave functions are normalized according to

$$\int d^3r\phi^*_{\vec{p}}(t,\vec{r}) i \overset{\leftrightarrow}{\frac{d}{dt}} \phi_{\vec{p'}}(t,\vec{r}) = (2\pi)^3\delta(\vec{p}-\vec{p}') \tag{3.27}$$

which is similar to usual normalization condition for continuous spectrum in non-relativistic quantum mechanics

$$\int d^3r \Psi^\dagger_{\vec{p}}(\vec{r},t)\Psi_{\vec{p}'}(\vec{r},t) = (2\pi)^3\delta(\vec{p}-\vec{p}') \tag{3.28}$$

The function (3.26) describes a relativistic particle with mass m. moving with the momentum $\vec{p}$.

3.2 Propagation amplitude of the free π-meson

Let us recall the non-relativistic case considered in Lecture I. The free non-relativistic propagation function was:

$$K_0^{NR}(x_2,x_1) = \int \frac{d^3p}{(2\pi)^3}\Psi_{\vec{p}}(x_2)\Psi^*_{\vec{p}}(x_1) \tag{3.29}$$

where

$$\Psi_{\vec{p}}(x) = e^{-i\frac{\vec{p}^2}{2m}t + i\vec{p}\vec{r}} \qquad\qquad x = (t,\vec{r}) \tag{3.30}$$

is the non-relativistic plane wave with momentum $\vec{p}$. The meaning of the propagation function is that it gives the "time evolution" of the wave function, namely

$$\Psi(x_2) = \int d^3r_1 K^{NR}(x_2,x_1)\Psi(x_1) \tag{3.31}$$

so for the non-relativistic plane waves this reduces to

$$\Psi_{\vec{p}}(x_2) = \int d^3r_1 K_0^{NR}(x_2,x_1)\Psi_{\vec{p}}(x_1) \tag{3.32}$$

since the momentum of the plane wave is conserved in the non-interacting theory. In the relativistic theory we shoud expect the same result, namely that the non-interacting plane wave should remain intact. In order to write down this property of the relativistic plane wave, we note that for the non-relativistic case the formula (3.32) is a direct consequence of orthogonality of plane waves:

$$\begin{aligned}\Psi_{\vec{k}}(x_2) &= \int \frac{d^3p}{(2\pi)^3}\Psi_{\vec{p}}(x_2)(2\pi)^3\delta(\vec{k}-\vec{p}) = \int \frac{d^3p}{(2\pi)^3}\Psi_{\vec{p}}(x_2)\int d^3r_1 \Psi^*_{\vec{p}}(x_1)\Psi_{\vec{k}}(x_1) = \\ &= \int d^3r_1 \left(\int \frac{d^3p}{(2\pi)^3}\Psi_{\vec{p}}(x_2)\Psi^*_{\vec{p}}(x_1)\right)\Psi_{\vec{k}}(x_1) = \int d^3r_1 K_0^{NR}(x_2,x_1)\Psi_{\vec{k}}(x_1)\end{aligned} \tag{3.33}$$

Let us now perform the same steps for the relativistic plane wave. Using the orthogonality condition for the relativistic plane waves given by eq. (3.27) one gets:

$$\begin{aligned}\phi_{\vec{k}}(x_2) &= \int \frac{d^3p}{(2\pi)^3}\phi_{\vec{p}}(x_2)(2\pi)^3\delta(\vec{k}-\vec{p}) = \int \frac{d^3p}{(2\pi)^3}\phi_{\vec{p}}(x_2)\int d^3r_1 \phi^*_{\vec{p}}(x_1)\, i\,\frac{\overleftrightarrow{d}}{dt_1}\,\phi_{\vec{k}}(x_1) = \\ &= \int d^3r_1 \left(\int \frac{d^3p}{(2\pi)^3}\phi_{\vec{p}}(x_2)\phi^*_{\vec{p}}(x_1)\right) i\,\frac{\overleftrightarrow{d}}{dt_1}\,\phi_{\vec{k}}(x_1)\end{aligned} \tag{3.34}$$

The last line in this equation can be rewritten in a form similar to the eq. (3.32) [10] :

$$\phi_{\vec{p}}(x_2) = \int d^3r_1 K_0(x_2, x_1) i \overset{\leftrightarrow}{\frac{d}{dt}} \phi_{\vec{p}}(x_1) \tag{3.35}$$

where the propagation amplitude for the free plane wave has the form:

$$K_0(x_2, x_1) = \int \frac{d^3p}{(2\pi)^3} \phi_{\vec{p}}(x_2)\phi^*_{\vec{p}}(x_1) \tag{3.36}$$

Moreover, since in the non-interacting theory the wavefunction of a relativistic particle in an arbitrary state can be expressed as a superposition of plane waves (see eq. (3.8)) the formula (3.35) holds for the time evolution of any wavefunction of a free particle.

Let us find an explicit form of K_0. We have:

$$K_0(t_2, \vec{r}_2, t_1, \vec{r}_1) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-iE_p(t_2-t_1)+i\vec{p}(\vec{r}_2-\vec{r}_1)} \tag{3.37}$$

It is convenient to rewrite the propagation function (3.37) in the relativistic-invariant form. Using the formula (1.4) for the δ-function it is easy to show that

$$K_0(x_2, x_1) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x_2-x_1)} 2\pi\delta(p^2 - m^2)\Theta(p_0) \tag{3.38}$$

reduces to the (3.37) after the integration over p_0 [11]. The propagation function (3.38) satisfies the Klein-Gordon equation:

$$(\Box_2 + m^2)K(x_2, x_1) = 0 \tag{3.39}$$

(where $\Box_2 \equiv \frac{d^2}{dt_2^2} - \frac{d}{dx_{2i}}\frac{d}{dx_{2i}}$) with the initial condition

$$2i\frac{d}{dt_2} K(x_2, x_1)|_{t_2=t_1} = (2\pi)^3\delta(\vec{r}_2 - \vec{r}_1) \tag{3.40}$$

Part V

3.3 Propagation of the π-meson in the external field

Let us study the interaction of our relativistic particle with the external field described by a certain potential $V(x)$. In the non-relativistic case the interaction with the potential was described by the set of diagrams in Fig. 15. In particular, the first nontrivial diagram gave

[10] Note that it would be wrong to expect the formula exactly of the same type as eq. (3.29) $\phi_{\vec{p}}(x_2) = \int d^3r_1 K_0(x_2, x_1)\phi_{\vec{p}}(x_1)$ since it would mean that in order to find the function ϕ at any given time it is sufficient to know this function at a certain time $t = t_1$. Even in the classical relativistic mechanics, it is not true: in order to find $\phi(x)$ it is nessesary to know not only the function itself at $t = t_1$ but also its first derivative. (Mathematically, the difference with the non-relativistic case is due to the fact that $\phi(x)$ obeys second-order differential equation (3.1) while Schrödinger equation is of the first order in time). Note that in our formula (3.35) the function $\phi(x)$ at time t_2 is determined by both $\phi(x)$ and $\frac{d}{dt}\phi(x)$ at time t_1.

[11] From this form it is easy to see that $K_0(x_2, x_1)$ corresponds to the propagation of plane waves with positive frequencies only - due to the factor $\Theta(p_0)$. The meaning of this property is that if we had a wave-function at some time $t = t_1$ constructed from positive frequencies the wave functions at later moments of time will also have only positive frequencies - otherwise the probabilistic interpretation would be impossible.

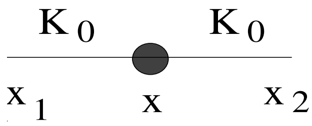


Figure 16. Non-relativistic propagation function in the first Born approximation

the contribution to the propagation function of the type

$$\int_{t_1}^{t_2} dt \int d^3r K_0^{NR}(x_2 - x)V(x)K_0^{NR}(x, x_1) \tag{3.41}$$

The time ordering was formalized by writing down Feynman rules in terms of Green functions

$$G_0^{NR}(x_2 - x_1) = \Theta(t_2 - t_1)K_0^{NR}(x_2 - x_1) \tag{3.42}$$

and it corresponded to causality: the non-relativistic particle was created in the point $\vec{r}_1$ at time $t = t_1$, then it interacted with the potential at time $t > t_1$ and finally it propagated to the point x_2 at time $t_2 > t$.

For the relativistic particle we can also try to write down the interaction with external potential (in the lowest order) of the form

$$\int dx K_0(x_2 - x)V(x)K_0(x - x_1) \tag{3.43}$$

but we face the problem that the condition $t_2 > t > t_1$ cannot be imposed in the relativistic invariant way. Indeed, the condition $t > t_1$ is meaningful only for the time-like intervals $(x - x_1)^2 > 0$. If the interval $x - x_1$ is spacelike ($(x - x_1)^2 < 0$) the condition $t > t_1$ is not invariant and it can happen that $t > t_1$ in one frame and $t < t_1$ in another. So, for the relativistic particle we must modify somehow the relation between propagation function and Green function: our first guess of the eq. (3.42) type is not relativistic invariant [12].

So, we were not able to satisfy both the conditions

1. Propagation function (in the extenal field) should contain only positive frequencies,
2. The interaction took place at the moment of time $t_2 > t > t_1$

Thus, we must sacrifice one of these conditions in order to build relativistic description of thew interaction of our particle with the external potential. To sacrifice (1) appears too high a price since we abandon the probabilistic interpretation of our wave functions. We shall try to modify the property (2) instead. Our problem is that

$$\int_{t_1}^{t_2} dt \int d^3r \; K_0(x_2 - x)V(x)K_0(x - x_1) \tag{3.44}$$

corresponding to the diagram in Fig.(16) is not relativistic invariant. Let us try to add smth to the r.h.s. of this equation "in a minimal way" so as to get a relativistic-invariant expression. It turns out that the following sum corresponding to the three diagrams in Fig. (17) is relativistic invariant:

[12] Everything would be OK with the relativistic invariance of the formula (3.42) if $K_0(x - y) = 0$ for spacelike intervals $x - y$ but we see from the explicit form of K_0 (3.37) that it is unfortunately not the case.

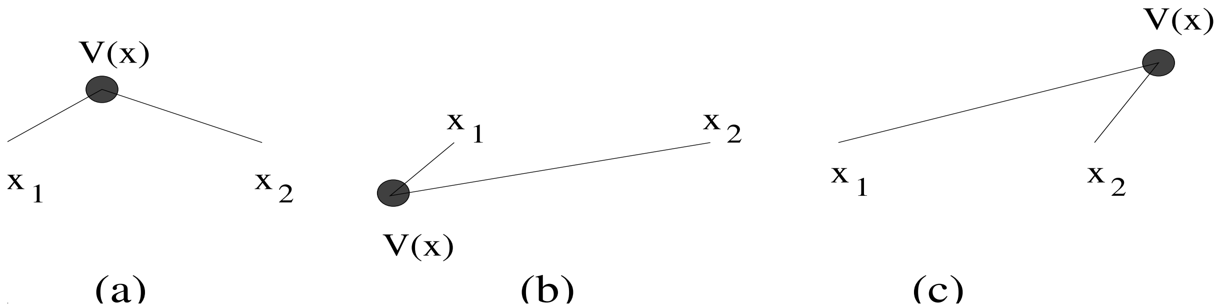


Figure 17.

$$\int_{t_1}^{t_2} dt \int d^3r\ K_0(x_2-x)V(x)K_0(x-x_1) + \int_{-\infty}^{t_1} dt \int d^3r\ K_0(x_2-x)V(x)K_0(x_1-x)$$
$$+ \int_{t_2}^{\infty} dt \int d^3r\ K_0(x-x_2)V(x)K_0(x-x_1) \tag{3.45}$$

Here the first term is (3.44) describes the situation when the particle was created in the point $\vec{r}_1$ at time $t=t_1$, propagated to the point $\vec{r}$ where at the the time t it interacted with the potential and finally it was absorbed in the point $\vec{r}_2$ at the time t_2. The interpretation of other two terms in r.h.s. of eq. (3.45) is differrent - see Fig. 17. The second term (Fig. 17b) corresponds to the situation when the external potential creates two particles at the moment of time $t < t_1, t_2$, they propagate (the propagation is described by our function K_0) and then one of them is annihilated in the point $\vec{r}_1$ at time t_1 and the second in $\vec{r}_2$ at time t_2. Similarly, the interpretation of the third term (see Fig. 17c) is as folows: the two particles were created at $t_1, \vec{r}_1$ and $t_2, \vec{r}_2$ and after that they propagated to the point $t, \vec{r}$ where they had been absorbed by the potential [13].

Now let us prove that the sum (3.45) is relativistic invariant. The best way to do this is to rewrite the r.h.s. of the eq. (3.45) in the following form:

$$\int d^4x \{K_0(x_2-x)\theta(t_2-t)+K_0(x-x_2)\theta(t-t_2)\}V(x)\{K_0(x-x_1)\theta(t-t_1)+K_0(x_1-x)\theta(t_1-t)\} \tag{3.46}$$

The sum of the two propagation functions (multiplied by Θ-functions) in braces in r.h.s. of this equation is actually relativistic invariant . Indeed, using the explicit form of the free propagation function K_0 (3.37) it is easy to show that

$$\begin{aligned} K_0(x)\theta(t) + K_0(-x)\theta(-t) = \\ \int \tfrac{d^3p}{(2\pi)^3}\tfrac{1}{2E_p}(\theta(t)e^{-iE_pt+i\vec{p}\vec{r}} + \theta(-t)e^{-iE_pt+i\vec{p}\vec{r}}) \\ = \lim_{\epsilon\to 0}\int \tfrac{d^4p}{(2\pi)^4 i}\tfrac{1}{m^2-p^2-i\epsilon}e^{-ipx} \end{aligned} \tag{3.47}$$

The last transition here needs explanation which we give below. Let us consider the integral

[13] Such interpretation does not contradict conservation of energy since due to the Heisenberg uncertainity relation $\Delta E \Delta t \sim 1$ we can create any number of particles for a short time.

over p_0 in the r.h.s. of eq. (3.47):

$$\int dp_0 \frac{-e^{-ip_0 t}}{p_0^2 - |\vec{p}|^2 - m^2 + i\epsilon} = \int dp_0 \frac{-e^{-ip_0 t}}{(p_0 - E_p + i\epsilon)(p_0 + E_p - i\epsilon)} \tag{3.48}$$

It is easy to see that at time $t > 0$ one can close the contour in the lower half of the complex p_0 plane (see Fig. (18)) so the result will be given by the residue in the right pole. On the

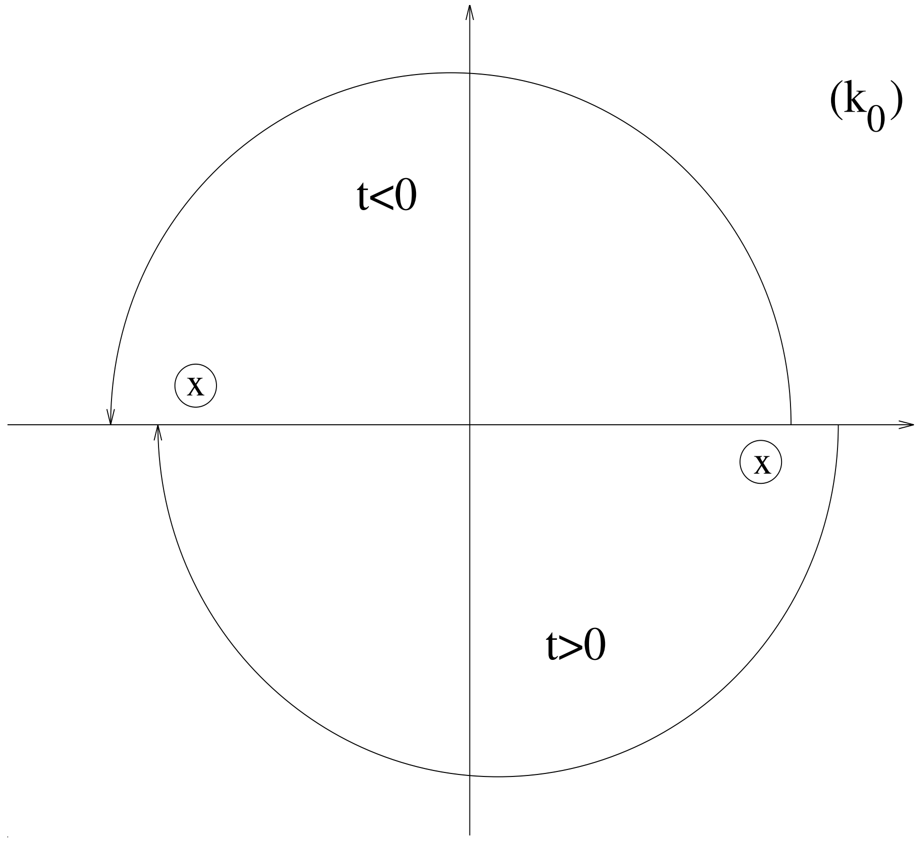


Figure 18. Calculation of the integral over p_0

other hand, at $t < 0$ one can add a contour in the upper half of the plane so the resulting contour integral will be given by the residue in the left pole. We get

$$\int dp_0 \frac{e^{-ip_0 t}}{p_0^2 - E_p^2 + i\epsilon} = -\frac{2\pi i}{2E_p} e^{-iE_p t}\theta(t) - \frac{2\pi i}{2E_p} e^{iE_p t}\theta(-t) \tag{3.49}$$

Now, substituting the eq. (3.49) in the r.h.s.of eq. (3.47) we obtain the l.h.s. of this equation. The expression in r.h.s. of this equation is called Feynman Green function of our scalar particle [14].

[14] The second name "Feynman" reflects the fact that the poles in the integral over k_0 are located as shown in Fig. (18) which ensures that the Green function can be expressed in the sum of two propagation functions (3.47). For any other choice of the locations of the poles (say, if both of them are above the axis) the Green function has both positive- and negative-frequency parts propagating in time which does not allow the probability interpretation (this was realized by Feynman).

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Indeed, it is easy to see that this function

$$G_0(x) = \int \frac{d^4p}{(2\pi)^4 i} \frac{1}{m^2 - p^2 - i\epsilon} e^{-ipx} \tag{3.50}$$

satisfies the inhomogenuous Klein-Gordon equation

$$(\Box + m^2) G_0(x) = \int \frac{d^4p}{(2\pi)^4 i} e^{-ipx} \frac{m^2 - p^2}{m^2 - p^2} = -i\delta^{(4)}(x) \tag{3.51}$$

so it is indeed the Green function of the KG equation in the mathematical sense.

We see now that the integrand in the r.h.s of eq. (3.45) is relativistic invariant and the integration goes over the whole space so the result is also invariant. It can be described by one Feynman diagram shown in Fig. (16) where each line corresponds now to the Feynman Green function:

$$\int d^4x G_0(x_2 - x) V(x) G_0(x - x_1) \tag{3.52}$$

This is the first-order correction to the Green function of the relativistic particle in the external potential $V(x)$. The total Green function is given by the sum of the diagrams shown in Fig.(19) just as in the non-relativistic case, only each line now corresponds to the

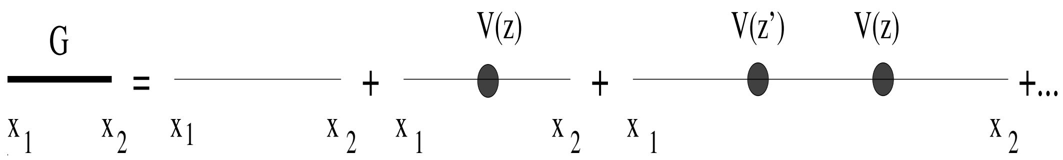


Figure 19. Feynman diagrams for the interaction of the relativistic particle with the potential

Feynman Green function (3.50) so that

$$\begin{aligned} & G(x_2, x_1) \\ = \;& G_0(x_2 - x_1) - i \int d^4z G_0(x_2 - z) V(z) G_0(z - x_1) \\ - & \int d^4z G_0(x_2 - z) V(z) G_0(z - z') V(z') G_0(z' - x_1) + ... \end{aligned} \tag{3.53}$$

This Green function satisfies the equation:

$$(\Box_2 + V(x_2)) G(x_2, x_1) = -i\delta^{(4)}(x_2 - x_1) \tag{3.54}$$

which describes ppropagation of the relativistic scalar meson through the external field $V(x)$.

In the momentum representation the set of Feynman rules will be the same as for the non-relativistic particle (see Lecture I and Fig. 15), only instead of the non-relativistic propagator $(\frac{p^2}{2m} - p_0 - i\epsilon)^{-1}$ we should write we should write down the Feynman propagator in the momentum space

$$\mathcal{G}_0(p) = \frac{1}{m^2 - p^2 + i\epsilon} \tag{3.55}$$

As in the non-relativistic theory, the transition matrix $T(p_1,p_2)$ is obtained by the "amputation" of Green function $G(p_1,p_2)$ ($\equiv$ removing the factors $\mathcal{G}_0(p_2)$,$\mathcal{G}_0(p_1)$, see eq. (2.56)) and the transition matrix "on the mass shell" $T(\vec{p}_1,\vec{p}_2)$ which determines the S-matrix element according to eq. (2.60) is obtained by the formula similar to eq. (2.61):

$$T(\vec{p}_1,\vec{p}_2) = T(p_1,p_2)|_{p_{10}=E_{p_1},p_{20}=E_{p_2}} \tag{3.56}$$

and the cross section is obtained in the same way as in lecture 3 (the only difference with the non-relativistic case will be the trivial factors due to the different relation between energy and momentum - for example, it will change the formula for the flux of initial particles)

It is instructive to get back the non-relativistic description for the case of small velocities $v \ll 1 \equiv |\vec{p}| \ll m$. Suppose $x_{20} > x_{10}$. For the free Green function we get

$$G_0(x_2,x_1) = K_0(t_2,\vec{r}_2,t_1,\vec{r}_1) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-iE_p(t_2-t_1)+i\vec{p}(\vec{r}_2-\vec{r}_1)} \tag{3.57}$$

At small velocities $E_p \simeq m + \frac{|\vec{p}|^2}{2m}$ so

$$\begin{aligned} G_0(x_2-x_1) &\simeq \frac{e^{-imt_{21}}}{2m}\int\frac{d^3p}{(2\pi)^3}e^{-i\frac{|\vec{p}|^2}{2m}(t_2-t_1)+i\vec{p}(\vec{r}_2-\vec{r}_1)} = \\ &= \frac{e^{-imt_{21}}}{2m}K_0^{\rm NR}(x_2-x_1) = \frac{e^{-imt_{21}}}{2m}G_0^{\rm NR}(x_2-x_1) \end{aligned} \tag{3.58}$$

In the first nontrivial order the Green function is given by the sum of three expressions in the r.h.s. of eq. (3.45). In the limit of small velocities the first term gives

$$\begin{aligned} &\int_{t_1}^{t_2}\int d^3x K_0(x_2-x)V(x)K_0(x-x_1) = \\ &= \int_{t_1}^{t_2}\int d^3x\frac{e^{-im(t_2-t)}}{2m}K_0^{\rm NR}(x_2-x)V(x)\frac{e^{-im(t-t_1)}}{2m}K_0^{\rm NR}(x-x_1) \\ &= \frac{e^{-imt_{21}}}{2m}\int_{t_1}^{t_2}\int d^3x G_0^{\rm NR}(x_2-x)\frac{1}{2m}V(x)G_0^{\rm NR}(x-x_1) \end{aligned} \tag{3.59}$$

The last two terms in r.h.s. of eq. (3.45) are small in the NR limit. For example, the second term reduces to

$$\begin{aligned} &\int_{-\infty}^{t_1}\int d^3x K_0(x-x_2)V(x)K_0(x-x_1) = \\ &= \frac{e^{-imt_{21}}}{2m}\int_{-\infty}^{t_1}\int d^3x K_0^{\rm NR}(x-x_2)\frac{e^{-2im(x_{10}-t)}}{2m}V(t,\vec{r})K_0^{\rm NR}(x_1-x) \end{aligned} \tag{3.60}$$

In the NR situation the characteristic scale of the potential is $\gg \frac{1}{m}$ so

$$\int_{-\infty}^{t_1} e^{-2im(t_1-t)}V(t,\vec{r}) \simeq \frac{1}{2m}V(t_1,\vec{r}) \tag{3.61}$$

which is small compared to the contribution (3.59) of the first term in Eq. (3.45). For example, in the case of the time-independent potential the first term in eq. (3.45) is proportional to

$$\int_{t_1}^{t_2} e^{-it\left(\frac{|\vec{p}_2|^2}{2m}-\frac{|\vec{p}_1|^2}{2m}\right)}V(\vec{r}) \simeq \frac{1}{\frac{|\vec{p}_2|^2}{2m}-\frac{|\vec{p}_1|^2}{2m}}V(\vec{r}) \tag{3.62}$$

so (3.61) $= O(|\vec{p}|^2/m^2) \otimes$ (3.62).

Similarly, one can prove that in general

$$G(x_2, x_1) = \frac{e^{-imt_{21}}}{2m} G^{\mathrm{NR}}(x_2, x_1) \tag{3.63}$$

where $G^{\mathrm{NR}}(x_2, x_1)$ is the non-relativistic Green function for the potential $\frac{1}{2m}V(x)$.

Part VI

4 Interactions of scalar mesons: the "πM model"

4.1 Interactions of particles in relativistic theory

At first let us recall how we write down Feynman diagrams for the two-particle interaction in the non-relativistic theory. The Schrödinger equation for the two-particle interaction in the non-relativistic quantum mechanics has the form [15]:

$$\frac{id}{dt}\Psi(t, \vec{r}, \vec{r}') = \left(-\frac{\nabla^2}{2m} - \frac{\nabla'^2}{2m} + V(\vec{r} - \vec{r}') \right) \Psi(t, \vec{r}, \vec{r}') \tag{4.1}$$

where $\nabla_i \equiv \frac{d}{dr_i}$ and $\nabla'_i \equiv \frac{d}{dr'_i}$. Similarly to the case of one non-relativistic particle considered in Sect. 1 we can introduce the two-paticle propagation function which describes the time evolution of the solution of the two-particle Schrödinger equation (4.1):

$$\Psi(t_2, \vec{r}_2, \vec{r}'_2) = \int d^3r_1 \int d^3r'_1 K^{NR}(t_2, \vec{r}_2, \vec{r}'_2; t_1, \vec{r}_1, \vec{r}'_1)\Psi(t_1, \vec{r}_1, \vec{r}'_1) \tag{4.2}$$

This propagation function is an exact solution of Schrödinger eq. (4.1)) with the initial condition

$$K^{NR}(t_1, \vec{r}_1, \vec{r}'_1; t_1, \vec{r}_2, \vec{r}'_2) = \delta(\vec{r}_1 - \vec{r}'_1)\delta(\vec{r}_2 - \vec{r}'_2) \tag{4.3}$$

This initial condition will ensure that at $t_2 = t_1$ our solution in r.h.s. of eq (4.2) reduces to initial-state wave function $\Psi(t_1, \vec{r}_1, \vec{r}'_1)$ [16]. The (non-relativistic) Feynman diagrams for this propagation function have the form (see Fig. 20)

$$\begin{aligned} K^{NR}(t_2, \vec{r}_2, \vec{r}'_2; t_1, \vec{r}_1, \vec{r}'_1) = \\ K_0^{NR}(t_2 - t_1, \vec{r}_2 - \vec{r}_1) K_0^{NR}(t_2 - t_1, \vec{r}'_2 - \vec{r}'_1) + \\ \int_{t_1}^{t_2} dt \int d^3r d^3r' K_0^{NR}(t_2 - t, \vec{r}_2 - \vec{r}) K_0^{NR}(t_2 - t, \vec{r}'_2 - \vec{r}') \\ (-iV(\vec{r} - \vec{r}')) K_0^{NR}(t - t_1, \vec{r} - \vec{r}_1) K_0^{NR}(t - t_1, \vec{r}' - \vec{r}'_1) + ... \end{aligned} \tag{4.5}$$

[15] For the two interacting particles the potential can depend only on the separation of the particles $|\vec{r} - \vec{r}'|$ due to the homogeneity and isotropy of the 3-space.

[16] If we want to describe the scattering of two particles with initial momenta $\vec{p}_1$ and $\vec{p}'_1$ the initial state can be taken as a superposition of free plane waves:

$$\Psi(t_1, \vec{r}_1, \vec{r}'_1)\Big|_{t_1 \to -\infty} e^{-i\frac{|\vec{p}_1|^2}{2m} + i\vec{p}_1\vec{r}} e^{-i\frac{|\vec{p}_2|^2}{2m} + i\vec{p}_2\vec{r}} \tag{4.4}$$

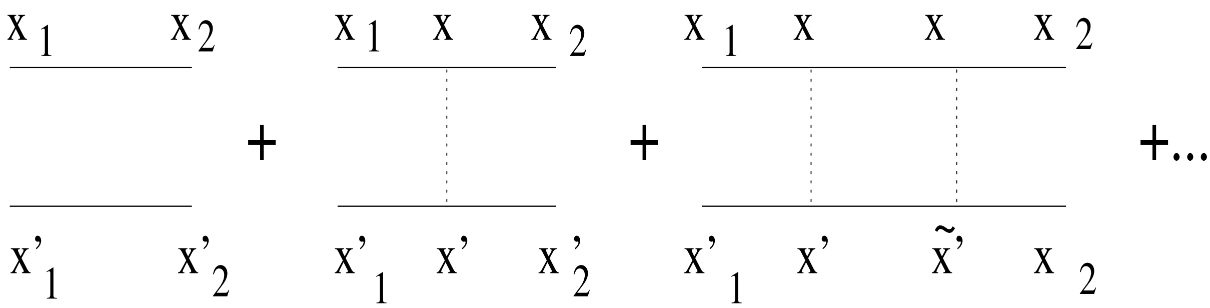


Figure 20. Non-relativistic Feynman diagrams for the two-particle interaction

It is easy to check that the series in r.h.s. of eq. (4.5) satisfies Schrödinger eq. (4.1) with the correct initial condition (4.3). In terms of Green function $G^{NR}(t_2, \vec{r}_2, \vec{r}_2'; t_1, \vec{r}_1, \vec{r}_1') \stackrel{\text{def}}{\equiv} \theta(t_2 - t_1)K^{NR}(t_2, \vec{r}_2, \vec{r}_2'; t_1, \vec{r}_1, \vec{r}_1')$ this series takes the form:

$$\begin{aligned} & G^{NR}(t_2, \vec{r}_2, \vec{r}_2'; t_1, \vec{r}_1, \vec{r}_1') = \\ & \qquad G_0^{NR}(t_2 - t_1, \vec{r}_2 - \vec{r}_1) G_0^{NR}(t_2 - t_1, \vec{r}_2' - \vec{r}_1') + \\ & \quad \int dt \int d^3r d^3r' G_0^{NR}(t_2 - t, \vec{r}_2 - \vec{r}) G_0^{NR}(t_2 - t, \vec{r}_2' - \vec{r}') \\ & (-iV(\vec{r} - \vec{r}')) G_0^{NR}(t - t_1, \vec{r} - \vec{r}_1) G_0^{NR}(t - t_1, \vec{r}' - \vec{r}_1') + \ldots \end{aligned} \tag{4.6}$$

The two-particle non-relativistic Green function (4.6) satisfies the inhomogenuous Schrödinger equation:

$$\left(\frac{id}{dt} + \frac{\nabla^2}{2m} + \frac{\nabla'^2}{2m} - V(\vec{r} - \vec{r}') \right) G^{NR}(t, \vec{r}, \vec{r}'; t_1, \vec{r}_1, \vec{r}_1') = \delta(t - t_1)\delta(\vec{r} - \vec{r}_1)\delta(\vec{r}' - \vec{r}_1') \tag{4.7}$$

The sum in r.h.s. of eq. (4.6) is given by the same diagrams in Fig. 20 with the lines being Green functions instead of the propagation amplitudes.

The interaction in non-relativistic quantum mechanics in instantaneous: the upper and lower points to which the potential is attached correspond to the same time. It is clear that such instantaneous interaction $V(|\vec{r} - \vec{r}'|)\delta(t - t')$ is not relativistic invariant . Instead of instantaneous interaction, we can try to invent some type of invariant potential like $V((x - x')^2)$. Since this potential depends only on the interval $(x - x')^2$ there will be no problem with relativistic invariance.

It turns out however that every non-local relativistic invariant potential corresponds to the exchange by a certain particle (or a group of particles) - in other words, each non-local interaction can be reduced to the local ones[17] (this is an experimental fact). In the simplest case, this is the exchange by one relativistic particle shown in Fig. 21.

For definiteness, suppose that the two scattered particles are scalar mesons with the mass M (let us call them M-mesons) which interact by exchange of π-mesons with mass m.

[17] The (hypotetical) example of the true non-local interaction which cannot be reduced to the local ones is studied in the so-called string theories

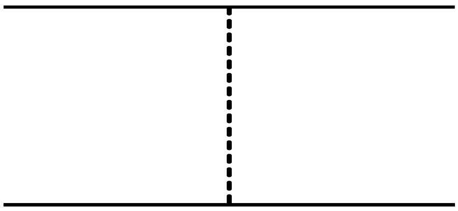


Figure 21. The two-particle interaction as an exchange by a third particle

The elementary process here is the emission of π-meson by the M-meson (see Fig. 22) so let us study this process first [18].

The amplitude for this emission (in the coordinate space) has the form:

$$G(x_2, x_3; x_1) = i \int d^4x N_0(x_2 - x) G_0(x_3 - x) \lambda N_0(x - x_1) \tag{4.8}$$

where G_0 is the free relativistic Green function of the π-meson (see eq. (3.50)) and N_0 so

$$N_0(x) = \int \frac{d^4p}{(2\pi)^4 i} \frac{1}{M^2 - p^2 - i\epsilon} e^{-ipx} \tag{4.9}$$

denotes the similar Green function for the M-meson. The quantity λ which describes the local interaction could, in principle, depend on the point x: $\lambda = \lambda(x)$ but due to the uniformity and homogenuety of the space it should be constant.

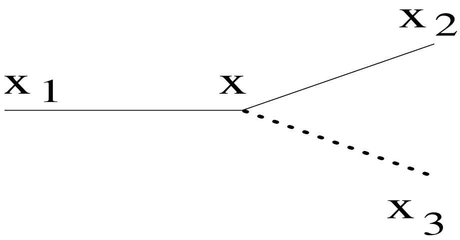


Figure 22. The emission of π-meson (dotted line) by M-meson (denoted by solid line)

If we substitute the explicit expressions for the Green functions (3.50) and (4.9) into the three-point Green function (4.8) we obtain:

$$\begin{aligned} G(x_2, x_3; x_1) &= \\ = i\lambda \int \frac{\bar{d}^4p_1}{i} \frac{\bar{d}^4p_2}{i} \frac{\bar{d}^4p_3}{i} & \frac{1}{(M^2 - p_1^2 - i\epsilon)(M^2 - p_2^2 - i\epsilon)(m^2 - p_3^2 - i\epsilon)} \int d^4x e^{-ip_2(x_2 - x) - ip_3(x_3 - x) - ip_1(x - x_1)} \\ &= \int \bar{d}^4p_1 \bar{d}^4p_2 \bar{d}^4p_3 \; e^{-ip_2x_2 - ip_3x_3 + ip_1x_1} G(p_1, p_2, p_3) \end{aligned} \tag{4.10}$$

where

$$G(p_1, p_2, p_3) = \frac{-\lambda(2\pi)^4 \delta(p_2 + p_3 - p_1)}{(M^2 - p_1^2 - i\epsilon)(M^2 - p_2^2 - i\epsilon)(m^2 - p_3^2 - i\epsilon)} \tag{4.11}$$

is the three-point Green function in the momentum representation (see Fig. 23). Here the

[18] Since the diagram below is relativistic it describes both the decay of M-meson into M-meson plus π-meson $M \Rightarrow M + \pi$, the recombination $M + \pi \Rightarrow M$, and the annihilation $MM \Rightarrow \pi$ (or $\pi \Rightarrow MM$) depending on the relation between times t_1,t_2, and t_3.

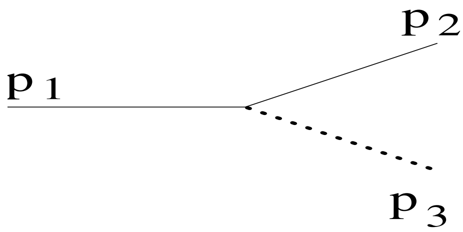


Figure 23. The Green function for the emission of π-meson by M-meson in the momentum representation

δ-function in the integrand stands for the conservation of momentum. For the future uses it is convenient to extract the δ-function and the factor -1 from the Green function (4.11) and define the reduced momentum-space Green function:

$$\mathcal{G}(p_1, p_2, p_3) = \frac{\lambda}{(M^2 - p_1^2 - i\epsilon)(M^2 - p_2^2 - i\epsilon)(m^2 - p_3^2 - i\epsilon)} \tag{4.12}$$

Does the Green function (4.10) describe the real decay of the M-meson? The answer is no, since we cannot satisfy all three conditions $p_1^2 = M^2$, $p_2^2 = M^2$, $p_3^2 = m^2$ if $p_1 = p_2 + p_3$. It is especially clear if we sit in the frame where this meson is at rest, so $p_1 = (M, 0, 0, 0)$. Then after the decay we have the same meson and π-meson moving one away from the other with some relative momentum $\vec{p}$. Therefore, the sum of the energies of these meson should be $\sqrt{M^2 + \vec{p}^2} + \sqrt{m^2 + \vec{p}^2} \geq M + m$, so this decay is impossible. However, it is possible for the short periods of time due to the Heisenberg uncertainity relation $\Delta E \Delta t = 1$ and therefore the Green function $G(x_1, x_2, x_3)$ given by eq. (4.10) is nonzero. But from the same uncertainity relation it follows also that the Green function should vanish at large times (greater than $\frac{1}{\Delta E} \sim \frac{1}{m}$). Let us demonstrate that

$$\lim_{t_1 \to -\infty, t_2, t_3 \to \infty} G(x_2, x_3; x_1) = 0 \tag{4.13}$$

At $t_1 \to -\infty, t_2, t_3 \to \infty$ we can close the contours of the integration over p_{10},p_{20}, and p_{30} in eq. (4.10) in the upper half-planes so

$$\begin{aligned}
&G(x_2, x_3 \to \infty; x_1 \to -\infty) \qquad (4.14)\\
&= \int đ^4p_1 đ^4p_2 đ^4p_3 \; e^{-ip_2x_2 - ip_3x_3 + ip_1x_1} \frac{-\lambda(2\pi)^4 \delta(p_2 + p_3 - p_1)}{(p_{10}^2 - M^2 - \vec{p}_1^2 + i\epsilon)(p_{20}^2 - M^2 - \vec{p}_2^2 - i\epsilon)(p_{30}^2 - m^2 - \vec{p}_3^2 + i\epsilon)}\\
&= \int đ^3p_1 đ^3p_2 đ^3p_3 \; \frac{i\lambda}{2E_{p_1} 2E_{p_2} 2E_{p_3}} e^{-iE_{p_2}x_{20} + i\vec{p}_2\vec{r}_2 - iE_{p_3}x_{30} + i\vec{p}_3\vec{r}_3 + iE_{p_1}x_{10} - i\vec{p}_1\vec{r}_1} (2\pi)^4 \delta(p_2 + p_3 - p_1)\\
&= \lambda \int đ^3p_2 đ^3p_3 e^{-ip_2(x_2 - x_1) - ip_3(x_3 - x_1)} \frac{2\pi\delta\Big(\sqrt{m^2 + \vec{p}_3^2} + \sqrt{M^2 + \vec{p}_2^2} - \sqrt{M^2 + (\vec{p}_2 + \vec{p}_3)^2}\Big)}{8\sqrt{m^2 + \vec{p}_3^2}\sqrt{M^2 + \vec{p}_2^2}\sqrt{M^2 + (\vec{p}_2 + \vec{p}_3)^2}}
\end{aligned}$$

As we demonstrated above, there is no point in the momentum space where the argument of the δ-function vanish so the result is zero.

Let us return now to the scattering of two M-mesons shown in Fig. 20. Since we know now the amplitude of the elementary process corresponding to the splitting (or recombination) of the M-meson we can write the amplitude of the scattering by exchange of one

π-meson . We have

$$G(x_2,x_2';x_1,x_1') = (i\lambda)^2 \int d^4z d^4z' N_0(x_2-z)N_0(z-x_1)G_0(z-z')N_0(x_2'-z')N_0(z'-x_1') \tag{4.15}$$

Using the explicit form of the free Green functions (3.50) and (4.9) we obtain

$$\begin{aligned} &G(x_2,x_2';x_1,x_1') \\ &= (i\lambda)^2 \int d^4z d^4z' \int \frac{đ^4p_1}{i}\frac{đ^4p_1'}{i}\frac{đ^4p_2}{i}\frac{đ^4p_2'}{i}\frac{đ^4p}{i} \\ &\times \frac{e^{-ip_1(z-x_1)-ip_2(x_2-z)-ip_1'(z'-x_1')-ip_2'(x_2'-z')-ip(z-z')}}{(M^2-p_1^2-i\epsilon)(M^2-p_1'^2-i\epsilon)(M^2-p_2^2-i\epsilon)(M^2-p_2'^2-i\epsilon)(m^2-p^2-i\epsilon)} \end{aligned} \tag{4.16}$$

The integration over z and z' yields two δ-functions which we can rewrite as follows:

$$\delta(p_2-p_1-p)\delta(p_1'-p_2'-p) = \delta(p_2-p_1-p)\delta(p_2+p_2'-p_1-p_1')) \tag{4.17}$$

The first of the δ-functions in the r.h.s. of this equation was used to perform the integration over p while the last one stands in the final answer and reflects the conservation of the momentum for the MM-scattering so we get

$$\begin{aligned} &G(x_2,x_2';x_1,x_1') \\ &= i\lambda^2 \int đ^4p_1 đ^4p_1' đ^4p_2 đ^4p_2' (2\pi)^4 \delta(p_1+p_1'-p_2-p_2') \\ &\times \frac{e^{-ip_1'x_1'-ip_2'x_2'+ip_1x_1+ip_2x_2}}{(M^2-p_1^2-i\epsilon)(M^2-p_1'^2-i\epsilon)(M^2-p_2^2-i\epsilon)(M^2-p_2'^2-i\epsilon)(m^2-(p_1-p_2)^2-i\epsilon)} \end{aligned} \tag{4.18}$$

It is convenient to rewrite Eq. (4.18) in terms of the momentum-space Green function:

$$\begin{aligned} &G(x_2,x_2';x_1,x_1') = \\ &\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_1'}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_2'}{(2\pi)^4} e^{ip_1'x_1'-ip_2'x_2'+ip_1x_1-ip_2x_2} G(p_2,p_2';p_1,p_1') \end{aligned} \tag{4.19}$$

where the Green function corresponding to the diagram in Fig. 24a has the form:

$$\begin{aligned} G_a(p_2,p_2';p_1,p_1') &= i(2\pi)^4\delta(p_1+p_1'-p_2-p_2')\mathcal{G}_a(p_2,p_2';p_1,p_1') \\ \mathcal{G}_a(p_2,p_2';p_1,p_1') &= \frac{\lambda^2}{(M^2-p_1^2-i\epsilon)(M^2-p_1'^2-i\epsilon)(M^2-p_2^2-i\epsilon)(M^2-p_2'^2-i\epsilon)(m^2-(p_1-p_2)^2-i\epsilon)} \end{aligned} \tag{4.20}$$

(here again we introduced the reduced Green function $\mathcal{G}$ with the δ-function excluded). We put label(a) here because except the diagram shown in Fig. (24a) there can be other possible processes shown in Fig. 24b and c.

In a similar way we can obtain the explicit expressions for contributions of the diagrams Fig. 24b and c. :

$$\begin{aligned} \mathcal{G}_b(p_2,p_2';p_1,p_1') &= \frac{\lambda^2}{(M^2-p_1^2-i\epsilon)(M^2-p_1'^2-i\epsilon)(M^2-p_2^2-i\epsilon)(M^2-p_2'^2-i\epsilon)(m^2-(p_1'-p_2)^2-i\epsilon)} \\ \mathcal{G}_c(p_2,p_2';p_1,p_1') &= \frac{\lambda^2}{(M^2-p_1^2-i\epsilon)(M^2-p_1'^2-i\epsilon)(M^2-p_2^2-i\epsilon)(M^2-p_2'^2-i\epsilon)(m^2-(p_1+p_1')^2-i\epsilon)} \end{aligned} \tag{4.21}$$

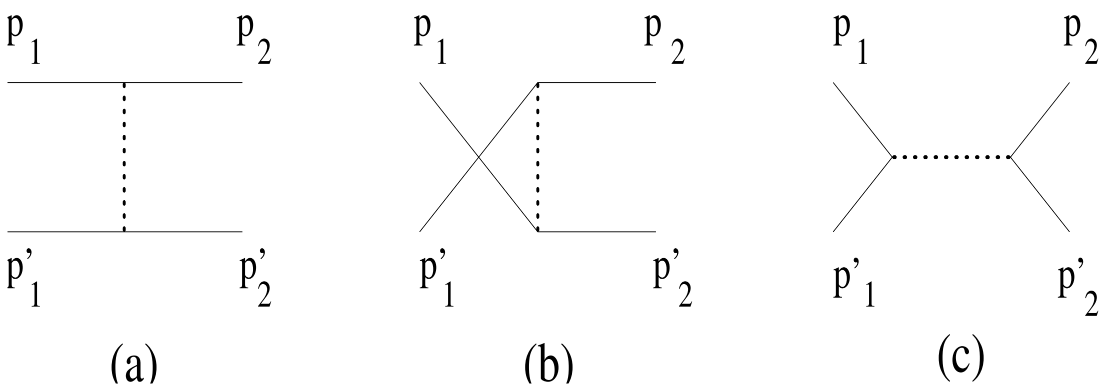


Figure 24. The Green function for the scattering of M-mesons by π-meson exchange in the momentum representation

(you may note that the only difference between these expressions stems from the different momentum flowing through the π-meson) so the total Green function of the $MM \to MM$ amplitude will be given by eq. (4.19) with

$$G(p_2, p_2'; p_1, p_1') = G_a(p_2, p_2'; p_1, p_1') + G_b(p_2, p_2'; p_1, p_1') + G_c(p_2, p_2'; p_1, p_1') \tag{4.22}$$

One may check that these are all possible diagrams for the MM scattering in the second order in λ -other diagrams as shown in Fig. 25 are simply the rewritings of the Fig. 24 in

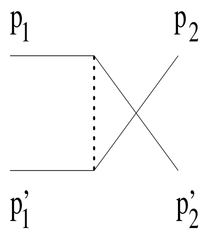


Figure 25. Another drawing of Fig. 11b diagram

a different way.

The diagram shown in Fig. 24c is called annihilation-type diagram since it describes not the scattering of two M-mesons by exchange of π-meson but rather the annihilation of the pair of M-mesons into π-meson with subsequent decay of the *virtual* π-meson into pair of M-mesons. Due to the relativistic invariance, the amplitude of this process must be described by the same constant λ^2 as the scattering process. Thus, the relativistic invariance predict some relations between the amplitudes of different processes and indeed these relations are confirmed experimentally.

Part VII

4.2 Feynman rules for "the πM theory"

We have learned that the propagation of the free massive particles (π-meson and M-meson in our πM model) is described by the Green functions (3.50) and (4.9) while the elementary process of the interaction - the emission of $M \Rightarrow M + \pi$ [19] is described by the elementary MMπ vertex which is $i\lambda$ where λ is a real number called the coupling constant for this model. Let us return to the scattering of two M-mesons. In previous Lecture we have considered the simplest case of the two elementary interactions and the resulting Green function for the scattering process was $\sim \lambda^2$. But since this elementary MMπ interaction can occur any number of times (at any place and at any time) we obtain an infinite set of possible Feynman diagrams of the type shown in Fig. (26):

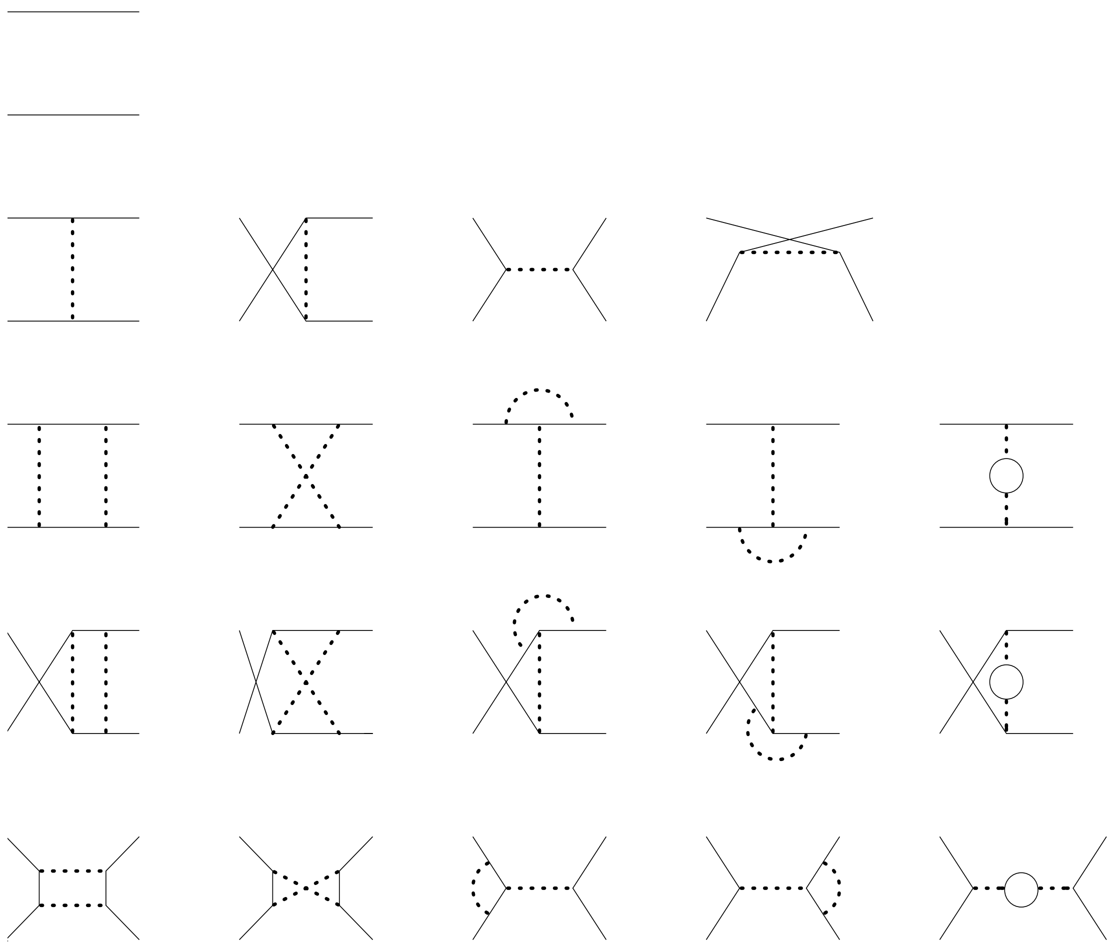


Figure 26. Typical Feynman diagrams for MM scattering

[19] or $M + \pi \Rightarrow M$, or $MM \Rightarrow \pi$, or $\pi \Rightarrow MM$ depending on the situation

Let us formulate carefully the Feynman rules for calculation of Green functions (in the coordinate space at first) in this πM theory. In order to draw all the diagrams for the Green function with m M-meson tails and l π-meson ones $G(x_1, x_2...x_m; y_1, y_2...y_l)$ one should perform the following steps:

1. Draw the $m + l$ end points (marking which of them correspond to M-mesons and which to π-mesons).

2. Draw any number (n) of πMM vertices. Each vertex comes with the factor $i\lambda/2$ and there is an integration over all the space over the position of each vertex.

3. Draw *all* possible connections between $m + l$ end points and n vertices. Each line will be the Green function G_0 (3.50) or N_0 (4.9) depending on the type of the line (better draw the different particles with different lines).

4. Divide the result by $n!$.

This $1/n!$ (and the factor $1/2$ in front of each vertex) are the combinatorial factors that will go away in the final answer (in some cases they do not go away entirely so it is better to keep trace of them). Let us illustrate how this works. Suppose we want to calculate the three-point MMπ Green function $G(x_1, x_2, y_1)$ in the third order in coupling constant. Using our rules (1)-(4) we draw the following picture

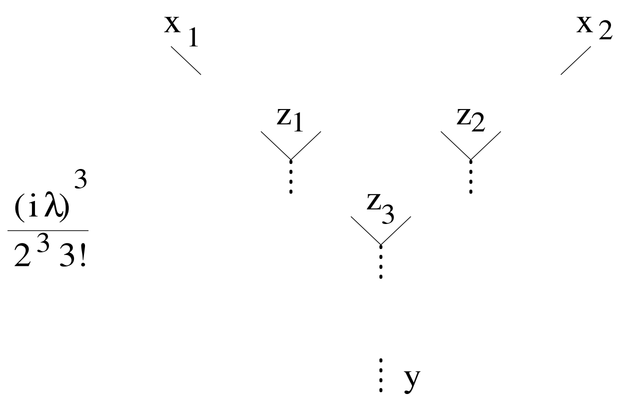


Figure 27. Feynman diagrams for MMπ transition in the λ^3 order: start

Now let us connect the end points and the vertices. Let us start with the point x_1. There are three possibilities : to connect it to z_1, z_2, or z_3. For each of these variants, there are two possibilities to connect x_1 with either left or right prong of the $M\pi M$ vertex. All of these 6 possiblities give the same result: it does not matter how we call the integration variable - z_1, z_2, z_3 (or even @). Let us call it, say, z_1. Thus, after the first step our result is given by Fig. 28.

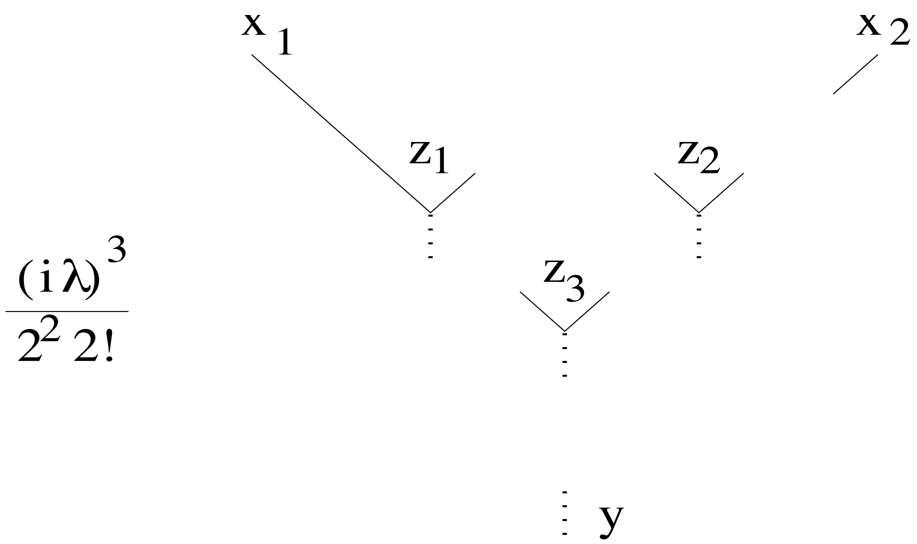


Figure 28. Feynman diagrams for MMπ transition in the λ^3 order: step 1

Consider now the point x_2. There are two possibilities: to connect it to the remaining tail at the point z_1 or to connect it to one of the points z_2, z_3. These to possibilities lead to different classes of diagrams. At first, we will consider the connection to z_1. There are no combinatorial factors at this step so we simply get

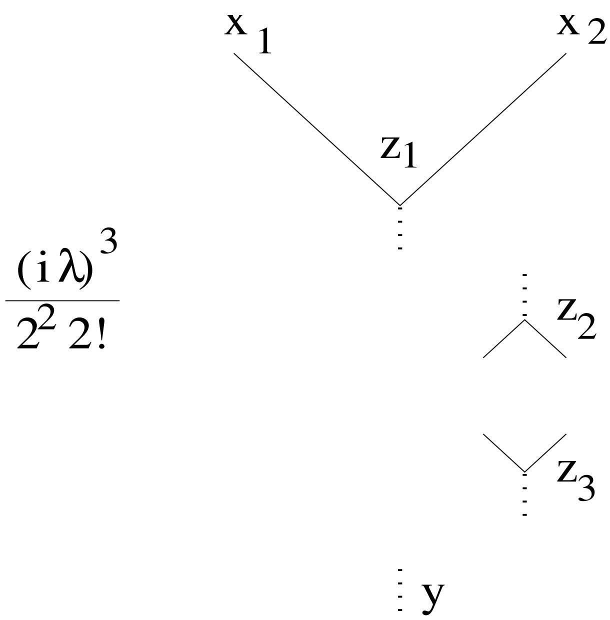


Figure 29. Feynman diagrams for MMπ transition in the λ^3 order: step 2, first set

Let us proceed with this picture and connect the point y. There are two ways to do this: (1) connect the point y to the remaining tail at the point z_1 (with no combinatorial factor) and, (2) connect y to either z_2 or z_3 (so the combinatorial factor is 2). The picture at this stage is given by Fig. 30

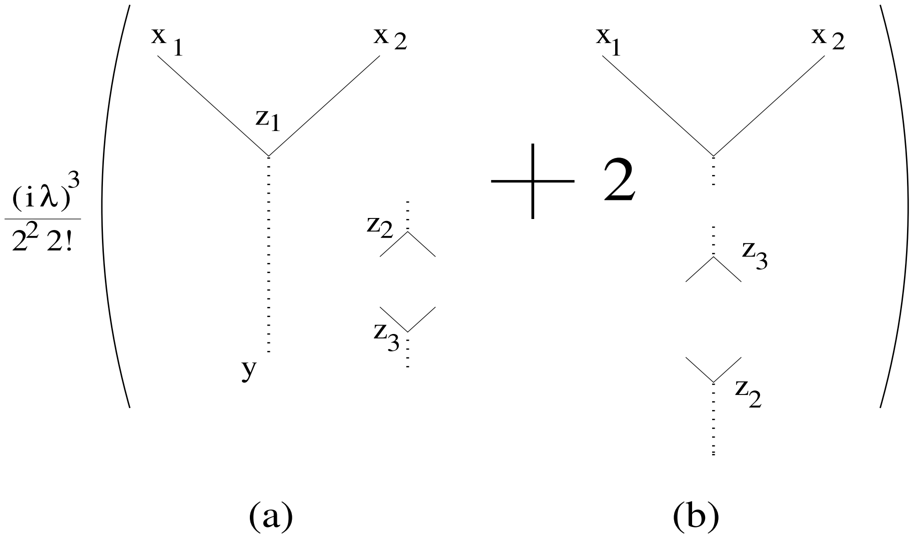


Figure 30. Feynman diagrams for MMπ transition in the λ^3 order: first set, step 3

The remaining step in both of this cases is obvious: to connect two π-meson lines (this brings no combinatorial factor) and to connect four remaining M-meson lines. If we connect meson lines from the same vertex the combinatorial factor is again 1 but if we connect M-mesons belonging to different vertices this factor is two (first time there is 2 M-mesons to choose from and second time there is only one). The final set of the diagrams for the first set is presented in Fig. 31

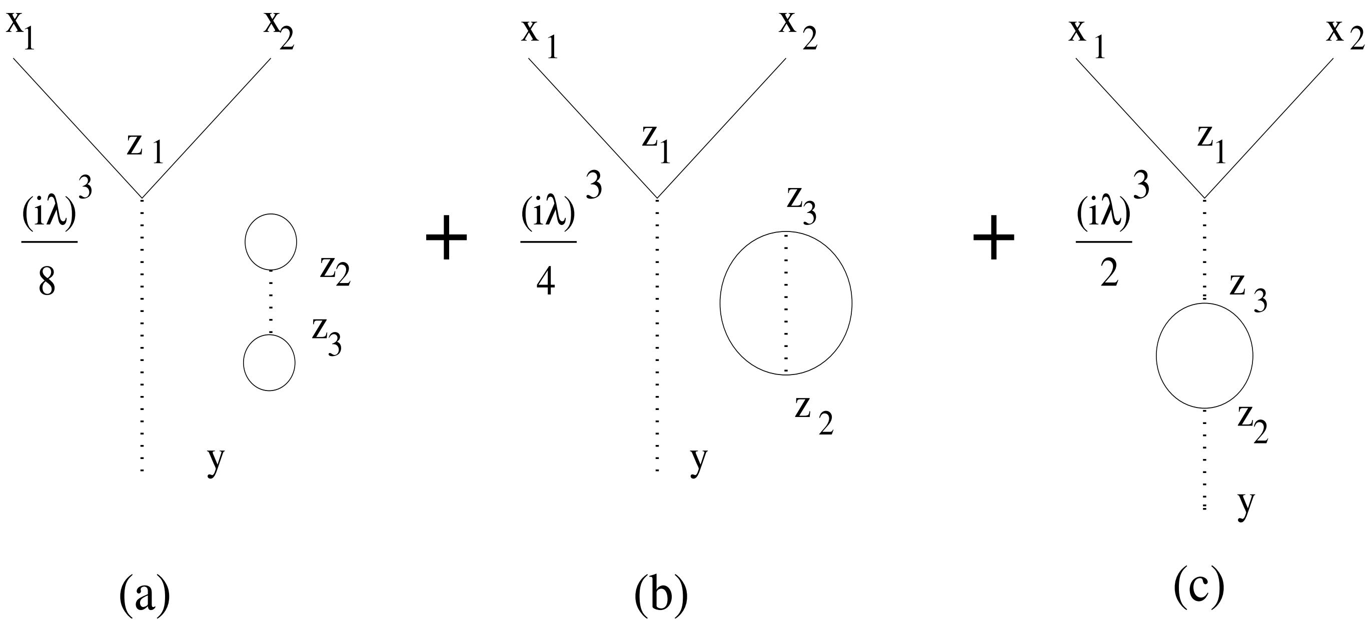


Figure 31. Feynman diagrams for MMπ transition in the λ^3 order: first set, finish

The first two diagrams are the examples of so-called disconnected diagrams which we will discuss (and throw away) in a minute.

Let us return to the step 1 and consider another possibility: the point x_2 is connected to one of the points z_2 or z_3. Since there are 2 of them to choose from and in both cases we can choose left or right M-meson the combinatorial factor is 4. Thus, at this step our picture is Fig. 32 (since the name of the integration variable is irrelevant, we called it z_2).

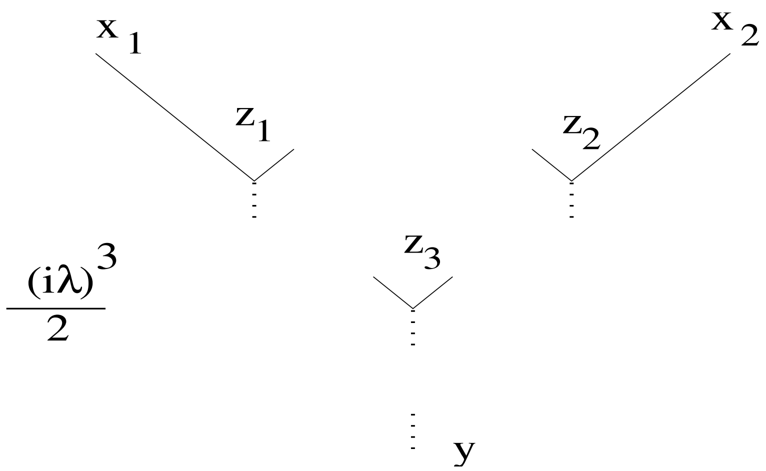


Figure 32. Feynman diagrams for MMπ transition in the λ^3 order: second set, step 2

Now let us connect at first the remaining M-meson lines. There are two ways to do this shown in Fig. 33a and b. The combinatorial factor for the (a) figure is 2 while for the (b) figure it is simply 1.

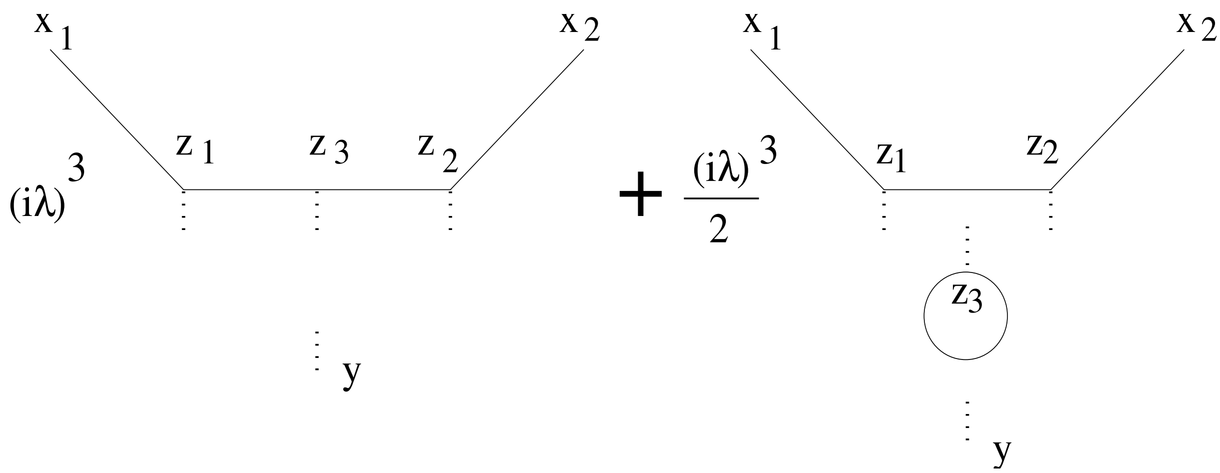


Figure 33. Feynman diagrams for MMπ transition in the λ^3 order: second set, step 3

Now we must connect the remaining π-meson lines. There are 3 possibilities for the diagram in Fig. 33a and two for the diagram in Fig. 33b so the final picture takes the form shown in Fig. 34 (there are no additional combinatorial factors at this step):

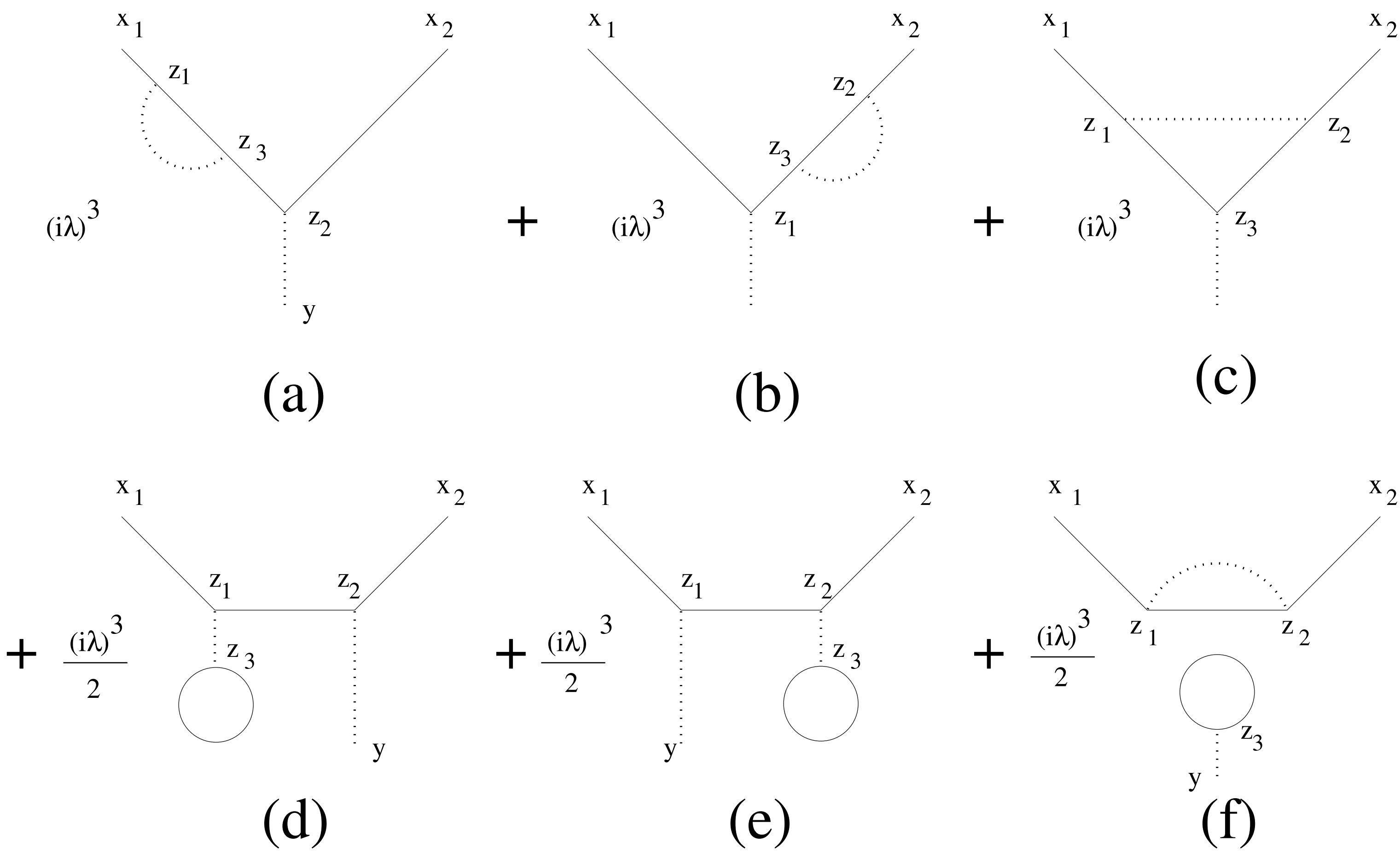


Figure 34. Feynman diagrams for MMπ transition in the λ^3 order: second set, finish

These diagrams and the diagrams in Fig. 31 form the complete set of Feynman diagrams for the $M \Rightarrow M + \pi$ transition in the third order in coupling constant. Each vertex comes with the factor $i\lambda$, each solid line is the free M-meson Green function (4.9), each dotted line is the π-meson Green function (3.50). You may note the factors like $1/2$ or $1/4$ in front of some diagrams. These are the remnants of the combinatorial factors in the numerator ($\equiv$ number of possible ways to connect lines) and in the denominator (the initial factor $\frac{1}{2^n n!}$, see Feynman rules **2** and **4**). Apart from these combinatorial factors, the rule is easy: simply draw all the possible diagrams. There actually exist the rule how to figure out the combinatorial factors (like $1/2$ in our example) just by inspection of the symmety group of the given diagram, but in simple cases it is easier to reobtain these factors from the rules **1-4**.

Part VIII

There are two types of Feynman diagrams: connected and disconnected. Let us compare the expressions for , say, diagram in Fig. 31b and Fig. 31c:

$$G(x_1, x_2; y)_b = \tag{4.23}$$
$$-i\frac{\lambda^3}{4}\left(\int dz_1 N_0(x_2 - z_1)N_0(z_1 - x_1)G_0(z_1 - y)\right)\left(\int dz_2 dz_3 [N_0(z_2 - z_3)]^2 G_0(z_2 - z_3)\right)$$
$$G(x_1, x_2; y)_c = \tag{4.24}$$
$$-i\frac{\lambda^3}{2}\int dz_1 dz_2 dz_3 N_0(x_2 - z_1)N_0(z_1 - x_1)G_0(z_1 - z_3)[N_0(z_3 - z_2)]^2 G_0(z_2 - y)$$

The integration over $z_2 + z_3$ in the second parentheses in r.h.s. of eq (4.23) is unrestricted - it gives the total volume VT of the 4-space so

$$G(x_1, x_2; y)_b = \left(i\lambda\int dz_1 N_0(x_2 - z_1)N_0(z_1 - x_1)G_0(z_1 - y)\right)\left(-\frac{\lambda^2}{4}VT\int dz_{23}[N_0(z_{23})]^2 G_0(z_{23})\right) \tag{4.25}$$

Just for comparison note that there are no such unrestricted integrations in the contribution (4.24) of the connected diagram in Fig.31. It is easy to note that the first factor in parentheses in the r.h.s. of eq. (4.23) is actually our Green function $G(x_1, x_2; y)$ in the first order in coupling constant (see eq. (4.8) in the previous Lecture). The second factor is the volume of the space-time multiplied by some number (the result of integration over z_{23}). Such factors are called vacuum bubbles. It will be demonstrated below that these vacuum bubbles exponentiate:

$$\sum \text{all vacuum bubbles} = e^{-iVT(\sum \text{connected vacuum bubbles})} \tag{4.26}$$

and therefore

$$\sum \text{all Green functions} = \left(\sum \text{connected Green functions}\right) * e^{-iVT(\sum \text{connected vacuum bubbles})} \tag{4.27}$$

This result has nothing to do with specifics of our Green functions and vertices – it is a general result valid for any diagram technique (another example is diagrams in statistical physics). Mathematically, it is a combinatorial property which follows from theory of graphs.

Let us illustrate this property. When we consider Feynman diagrams of the Fig. 31b type in higher odrers in perturbation theory our vacuum bubble can appear arbitrary number of times. After some algebra, one can see that the corresponding combinatorial factor is $1/n!$ where n is the number of bubbles. The expression for the contribution of this diagram with n bubbles of Fig. 31b type has the form:

$$G_{n\ bubbles}(x_1, x_2; y) = G^{(1)}(x_1, x_2; y)\frac{1}{n!}(-iVTB)^n \tag{4.28}$$

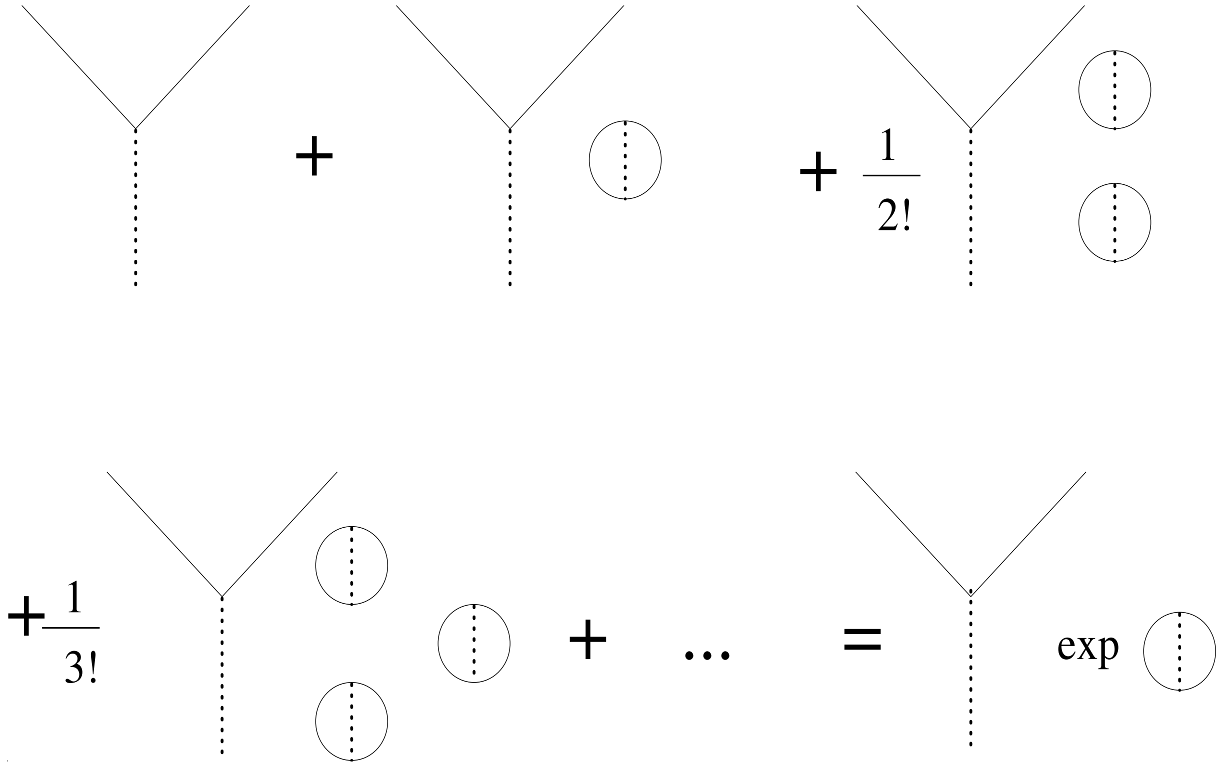


Figure 35. Exponentiation of vacuum bubbles

where

$$\begin{aligned} B &= -i\frac{\lambda^2}{4}\int dz_{23}[N_0(z_{23})]^2 G_0(z_{23}) \\ &= \frac{\lambda^2}{4}\int \frac{dp_1}{(2\pi)^4 i}\frac{dp_2}{(2\pi)^4 i}\frac{1}{(M^2-p_1^2-i\epsilon)(M^2-p_2^2-i\epsilon)(m^2-(p_1-p_2)^2-i\epsilon)} \end{aligned} \tag{4.29}$$

The (infinite) constant B has the dimension of an energy density. It represents the contribution of the vacuum bubble in Fig. (31b) to the shift of vacuum energy due to the interactions. The fact that this constant is real ($\Im B = 0$) can be proved using so-called Cutkovsky rules for imaginary parts of Feynman diagrams, see the discussion after Eq. (4.42) below.

From Eq. (4.28) one sees that the sum of vacuum bubbles exponentiates

$$G_{Fig.35} = G^{(1)} e^{-iVTB} \tag{4.30}$$

It can be demonstrated that this properyt is true for any bubble (and for the sum of any number of connected bubbles) so we get the exponentiation (4.27). The factor in the exponent in eq. (4.27), given by the sum of all connected vacuum bubbles, is the shift of vacuum energy due to the interaction. In the relativistic theory all our our Green functions $G(x_1, ...y_n)$ contain this factor $e^{-iE_{vac}T}$ (where $E_{vac} = V\epsilon_{vac}$ and ϵ_{vac} is the density of vacuum energy) [20] . This (infinite) factor is present in all amplitudes and has nothing to do with the scattering of the particles so it is convenient to exclude it from the definition of the Green functions for our transition amplitudes. So, the net result of our discussion is that there is no need to draw vacuum bubbles.

Apart from vacuum bubbles appearing in Fig. 31 there are other diagrams in Fig. 34 which may be disregarded. Let us discuss the so-called tadpole diagram shown in Fig.**??** This tadpole diagram does not actually depend on y so it is (an infinite) number which

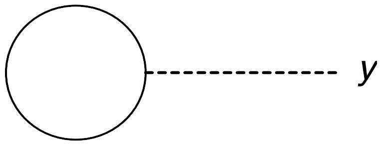


Figure 36. Tadpole diagram.

gives the average value of the classical π-meson field ϕ in the vacuum. (Intuitively, it is clear that if we allow the π-meson "disappear into vacuum" as shown in Fig. 36, there will be a large number of π-mesons in the vacuum and the large number of particles corresponds to the classical field). There are models with symmetry breaking where the average value of the scalar (Higgs) field in the vacuum is nonzero so the tadpole diagrams are allowed. We, however, will consider the simplest case of the theory without vacuum condensate of π-meson s. The "no vacuum condensate" property is translated into the Feynman diagram language as the "no tadpoles" requirement. In the case of gauge theories like QED the property that the tadpole diagrams vanish is preserved by gauge symmetry in each order in perturbation theory (we shall see it later). In our model we do not have any symmetry which ensures vanishing of the tadpoles so we must preserve it ourselves in each order in perturbation theory. To do this, we must redefine Green functions of our particles in such a way that there would be no tadpoles. Practically, we simply do not draw them. Therefore the set of the relevant Feynman diagrams for the $M \Rightarrow \pi + M$ transition reduces to Fig. 37

Apart from vacuum bubbles and tadpoles there are also disconnected diagrams with the two or more legs like shown in Fig. 38. This diagrams describe two independent processes going on in the space so their calculation will simply reproduce the product of amplitudes for their parts. Thus, these diagram bring no new information and we shall not bother to draw them.

Summarising, this gives us the additional Feynman rule

5 Draw only connected diagrams without tadpoles.

Now, the rules **1-5** give us the complete set of Feynman rules for our πM model.

4.3 Feynman rules in the momentum space

In previous Lecture we have formulated Feynman rules for the calculation of Green functions (in our πM model). Since the explicit form of the free Green function is much more simple

[20] It is worth noting that in the non-relativistic theory we cannot construct vacuum bubbles (although the shift of vacuum energy is still present).

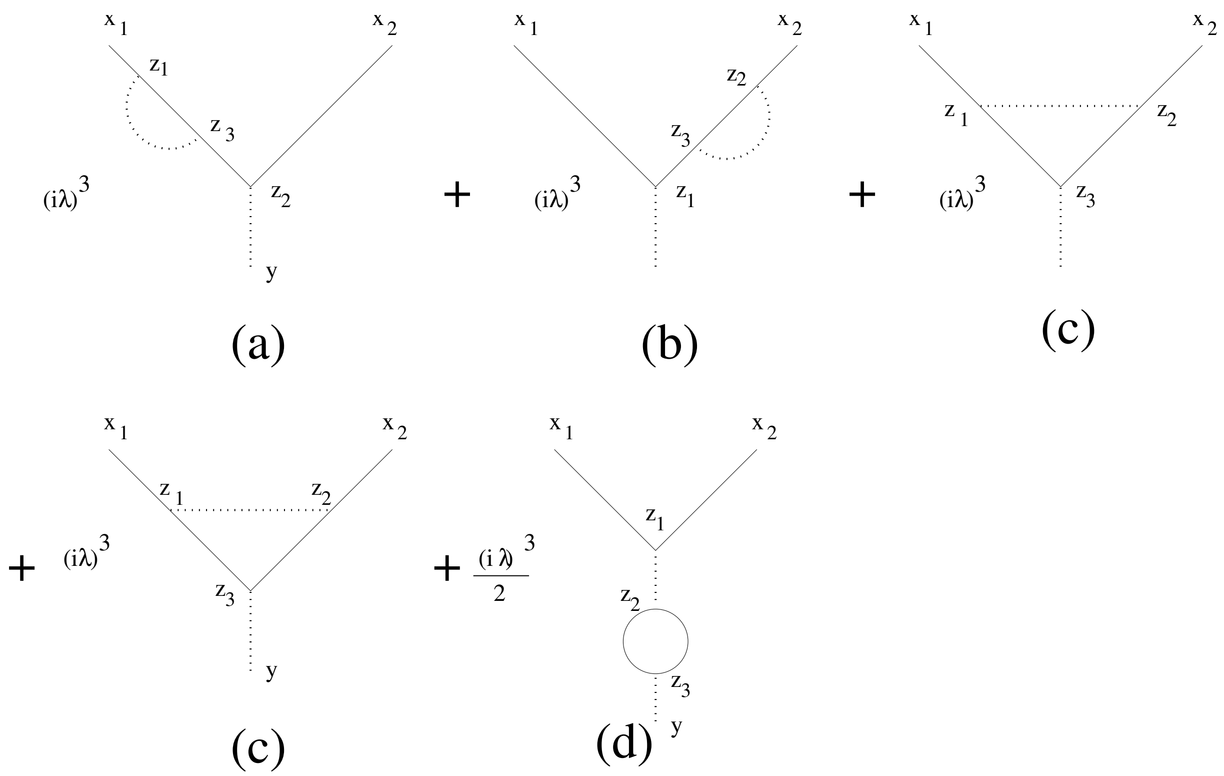


Figure 37. Connected diagrams without tadpoles for $M \Rightarrow \pi + M$ transition

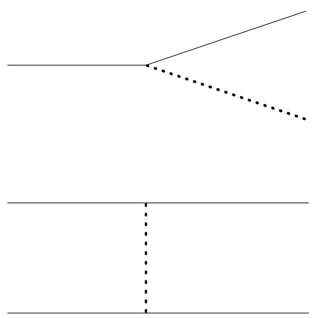


Figure 38. Disconnected diagram for $MMM \Rightarrow MMM\pi$ transition.

in the momentum space (see eq. (3.55)) one should expect that the explicit form of the Feynman diagrams should be simpler in the momentum representation. Let us consider the diagrams for $M \Rightarrow \pi + M$ transition which were studied in the Born approximation in Lecture 6. In the lowest order in perturbation theory ($\equiv$ in expansion in powers of λ) the corresponding three-point Green function has the form (4.10). In the next order in λ^2 this Green function is given by diagrams in Fig. 34. Let us consider the diagram in Fig. 34c as a typical example

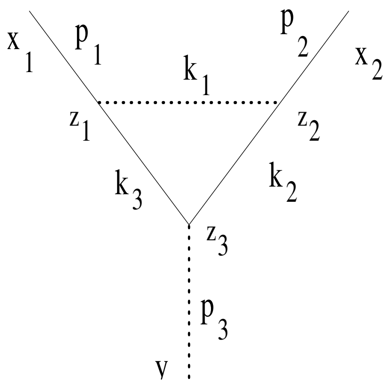


Figure 39. Typical one-loop diagram for the $\pi \Rightarrow \pi + M$ transition.

The corresponding three-point Green function has the form:

$$G(x_2, y; x_1) = \\ (i\lambda)^3 \int \tfrac{d^4p_1}{(2\pi)^4 i} \tfrac{d^4p_2}{(2\pi)^4 i} \tfrac{d^4p_3}{(2\pi)^4 i} \tfrac{1}{(M^2-p_1^2-i\epsilon)(M^2-p_2^2-i\epsilon)(m^2-p_3^2-i\epsilon)} \int d^4z_1 d^4z_2 d^4z_3 e^{-ip_2(x_2-z_2)-ip_3(y-z_3)-ip_1(z_1-x_1)} \\ \int \tfrac{d^4k_1}{(2\pi)^4 i} \tfrac{d^4k_2}{(2\pi)^4 i} \tfrac{d^4k_3}{(2\pi)^4 i} e^{-ik_2(z_3-z_2)-ik_3(z_1-z_3)-ik_1(z_2-z_1)} \tfrac{1}{(m^2-k_1^2-i\epsilon)(M^2-k_2^2-i\epsilon)(M^2-k_3^2-i\epsilon)} \tag{4.31}$$

It is easy to see that the integration over the position of each vertex gives the δ-function reflecting conservation of the momentum in this vertex, so we obtain:

$$G(x_2, y, x_1) = \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} e^{-ip_2x_2-ip_3y+ip_1x_1} G(p_1, p_2, p_3) \tag{4.32}$$

where the Green function in the momentum space is:

$$G(p_1, p_2, p_3) = \\ \tfrac{\lambda^3}{(M^2-p_1^2-i\epsilon)(M^2-p_2^2-i\epsilon)(m^2-p_3^2-i\epsilon)} \int \tfrac{\not{d}^4k_1}{i} \tfrac{\not{d}^4k_2}{i} \tfrac{\not{d}^4k_3}{i} \not{\delta}(k_1 - k_3 - p_1) \not{\delta}(k_2 - k_1 + p_2) \not{\delta}(k_3 - k_2 + p_3) \\ \tfrac{1}{(m^2-k_1^2-i\epsilon)(M^2-k_2^2-i\epsilon)(M^2-k_3^2-i\epsilon)} \tag{4.33}$$

One may check that this structure is universal: the set of Feynman rules for the Green function $G(p_1, p_2, ...p_n)$ in the momentum space is:
I. Draw all possible (but different!) diagrams with proper symmetry combinatorial factors.
II. Put $G_0(p) = \frac{1}{i(m^2-p^2-i\epsilon)}$ $\big($or $N_0(p) = \frac{1}{i(M^2-p^2-i\epsilon)}$ $\big)$ for each line with momentum p.
III. Put $-i\lambda(2\pi)^4\delta(\sum p_j)$ in each vertex (where p_j are the momenta flowing into this vertex).
IV. Integrate over the momenta of internal lines (an internal line is any line that is not the tail). Each integration over momenta comes with $(2\pi)^4$ in the denominator.

These Feynman rules can be simplified even more by performing the integration using these δ-functions. To demonstrate this, let us return to our 34c diagram and finish the calculation. We obtain:

$$\mathcal{G}(p_1, p_2, p_1 - p_2) = \\ \tfrac{\lambda^3}{(M^2-p_1^2-i\epsilon)(M^2-p_2^2-i\epsilon)(m^2-p_3^2-i\epsilon)} \int \tfrac{d^4k}{(2\pi)^4 i} \tfrac{1}{(m^2-k^2-i\epsilon)(M^2-(p_2-k)^2-i\epsilon)(M^2-(p_1-k)^2-i\epsilon)} \tag{4.34}$$

where the reduced Green function $\mathcal{G}(p_1, p_2, p_1 - p_2)$ was defined in previous Lecture according to

$$G(p_1, p_2, p_3) = -(2\pi)^4 \delta(p_1 - p_2 - p_3) \mathcal{G}(p_1, p_2, p_1 - p_2) \tag{4.35}$$

The expression for the reduced Green function (4.34) can be visualized as shown in Fig. 40

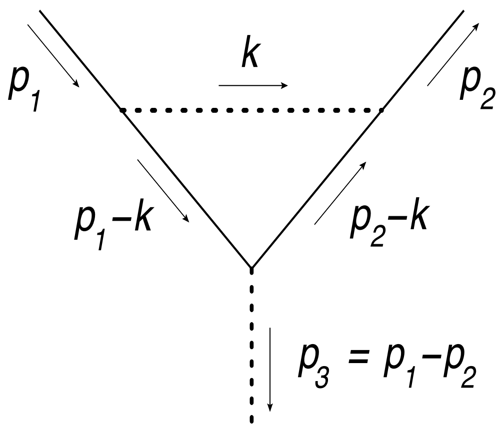


Figure 40. Momentum flow for our one-loop diagram.

We see, that after taking into account the momentum conservation in each vertex there is only one non-trivial integration (over k) corresponding to one *loop*. In more complicated diagrams, there are more loop integrals. On the other hand, if one cosnsiders the so-called *tree* diagrams ($\equiv$ without loops) the value of these diagrams in momentum representation is actually already fixed by simply drawing the diagram with taking into account the momentum conservation in each vertex. For example, the πM scattering diagrams shown in Fig. 24are given (in Born approximation) by the expressions (4.21).

Let us formulate the final set of rules for calculation of the reduced Green function in the momentum representation. First of all, the precise definition of $\mathcal{G}(p_1, p_2, ...p_n)$ is:

$$\begin{aligned} G(x_1, x_2, ...x_n) &= \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} ... \frac{d^4p_n}{(2\pi)^4} e^{-ip_1x_1 - ip_2x_2 ... - ip_nx_n} G(p_1, p_2, ...p_n) \\ G(p_1, p_2, ...p_n) &= (-i)^{n-1} (2\pi)^4 \delta(p_1 + p_2 + ... + p_n) \mathcal{G}(p_1, p_2, ...p_n) \end{aligned} \tag{4.36}$$

(Note that our definition of the Green function $\mathcal{G}_0(p) = (m^2 - p^2 - i\epsilon)^{-1}$ agrees with this general formula).

The Feynman rules for $\mathcal{G}(p_1, p_2, ...p_n)$ are:

I. Draw all different connected diagrams without tadpoles taking into account the symmetry (combinatorial) factors.

II. Draw momenta flow for each diagram taking into account conservation of the momentum in each vertex.

III. Each π-meson line with momentum p brings factor $\mathcal{G}_0(p) = \frac{1}{m^2 - p^2 - i\epsilon}$, each M-meson line - factor $\mathcal{N}_0(p) \equiv \frac{1}{M^2 - p^2 - i\epsilon}$, and each vertex - factor λ

IV. There is an integration $\int \frac{d^4k}{(2\pi)^4 i}$ for each loop.

Homework assignment 2:.

Draw Feynman diagrams for the two-π-meson Green function in order λ^4 (connected diagrams, no tadpoles). Write down the corresponding expressions for $\mathcal{G}$ in the momentum representation.

Part IX

4.4 General structure of Feynman diagrams

The that Green functions contain all the physical information about the theory - if you calculated all the Green functions, you know everything (except maybe how to get the cross sections from these Green functions - because this will be explained only in the next Lecture!). In general, the Green functions are classified according to number of tails and loops. For a given proicess, the number of tails ($\equiv$ end points) is fixed. For example, for the $MM\pi \Rightarrow MM\pi$ transition shown in Fig. 41 below there are four M-meson tails and two

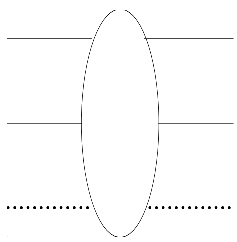


Figure 41. $MM\pi \Rightarrow MM\pi$ transition.

pion ones. When the number of tails in fixed, the Green functions are classified according to the number of *loops* - integrations over momenta. In the lowest order, there are no loops at all, and we get so-called *tree* diagrams, see Fig. 42a below. It is easy to see that each extra loop comes with the factor λ^2 so if λ is small, the tree diagrams are the most important ones and each extra loop bring additional smallness. The examples of diagrams with one and two loops for our example are shown in Fig. 42b and c.

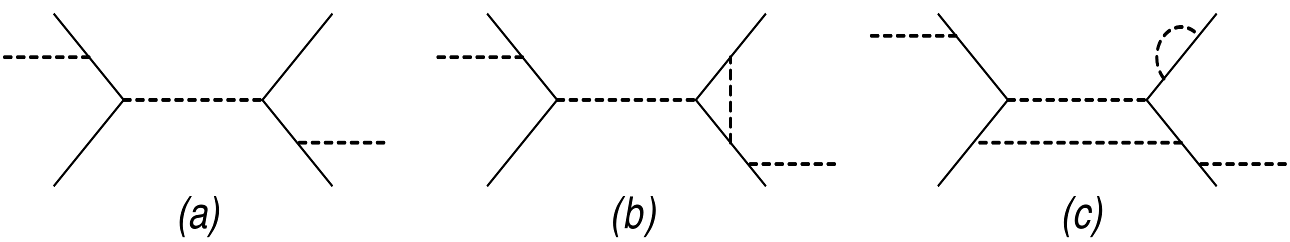


Figure 42. Examples of the tree (a), one-loop (b), and two-loop (c) diagrams for $MM\pi \Rightarrow MM\pi$ transition.

It is easy to check that in conventional units an extra loop brings also an additional factor $\hbar$, so the expansion in number of loops in the expansion in "quantumness" of our process.

It is easy to note that some of the integrals for the diagrams are divergent at large loop momenta (it is called "an UV divergency". The typical examples are the so-called self-energy corrections and vertex corrections shown for example in Fig.42b and c respectively. The analytical expression for this so called self-energy insertion is (see Fig. 43)

$$\mathcal{N}(p) = \frac{1}{M^2 - p^2 - i\epsilon}\left(\int \frac{d^4k}{(2\pi)^4 i}\frac{1}{(M^2-(p-k)^2-i\epsilon)(m^2-k^2-i\epsilon)}\right)\frac{1}{M^2-p^2-i\epsilon} \tag{4.37}$$

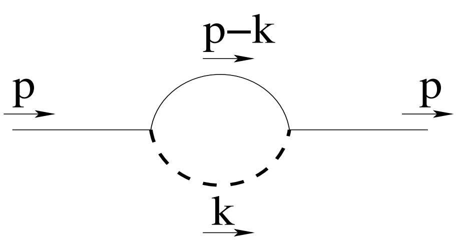


Figure 43. Self-energy diagram

It is easy to see that the integral over k in parenthesis is actually divergent at large k. Let us consider for simplicity the case $p^2 < 0$ and draw the position of the poles in the integral over k_0 in the complex k_0 plane. Let us take the frame where $p_0 = 0$ (you can always find such frame for the space-like 4-vector p. There are four poles

$$\begin{aligned} (k_0)_{1,2} &= \pm\sqrt{\vec{k}^2 + m^2} \\ (k_0)_{3,4} &= \pm\sqrt{(\vec{p}-\vec{k})^2 + M^2} \end{aligned} \tag{4.38}$$

which are located as shown in Fig. 45

The integration goes over the real axis, and it is easy to see that we can turn the contour of integration on 90^o counterclockwise. After that, we make the substitution $k_0 \to ik_0$ so the total integral over 4-momentum k will have the Euclidean form:

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(M^2+(p-k)^2)(m^2+k^2)} \tag{4.39}$$

Now it is obvious that it diverges at large k. In the same way one can check that the integral corresponding to the vertex correction is also divergent. These are the example of UV divergencies in the field theories. There is a special procedure called *renormalization* to deal with such divergencies.

We have seen that the integral

$$\int \frac{d^4k}{(2\pi)^4 i}\frac{1}{(M^2-(p-k)^2-i\epsilon)(m^2-k^2-i\epsilon)} \tag{4.40}$$

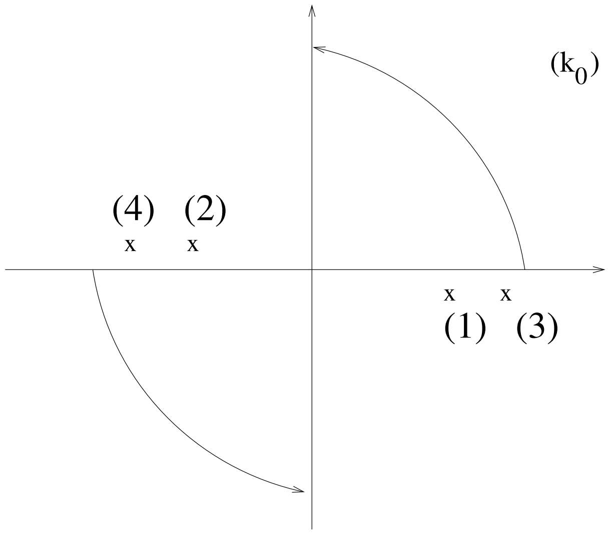


Figure 44. Singularities in complex k_0 plane for the self-energy integral at $p^2 < 0$.

is real at $p^2 < 0$. Actually, if we increase p^2 it will be real until $p^2 = (M+m)^2$ and then it will gain imaginary part corresponding to the creation of physical (on-mass-shell) particles. Indeed, at $p^2 > 0$ we can always find a frame where $\vec{p} = 0$ so the structure of the poles in k_0 plane is as shown Fig. 45

$$\begin{aligned} (k_0)_{1,2} &= \pm\sqrt{\vec{k}^2 + m^2} \\ (k_0)_{3,4} &= p_0 \pm \sqrt{\vec{k}^2 + M^2} \end{aligned} \tag{4.41}$$

It is clear that you can rotate the contour until p_0 is so large that the pole (4) is to the right of the pole (1) so the dashed contour in Fig. 45 will be pinched by these two poles when $p_0 \geq \sqrt{\vec{k}^2 + m^2} + \sqrt{\vec{k}^2 + M^2}$ which means that the energy p_0 is sufficient to produce two *physical* (on-shell) particles with momenta $\vec{k}$ and $-\vec{k}$. Since we have integration over $\vec{k}$ the first time it will happen at very small k so the condition of the imaginary part is $p_0 \geq M + m$ which translates to $p^2 \geq (M+m)^2$ in our frame. It can be demonstrated that this imaginary part can be written as

$$\begin{aligned} 2i\Im \int \frac{d^4k}{(2\pi)^4 i} &\frac{1}{(M^2 - (p-k)^2 - i\epsilon)(m^2 - k^2 - i\epsilon)} \\ &= \int \frac{d^4k}{(2\pi)^4 i} 2\pi i\delta((p-k)^2 - M^2)\theta(p_0 - k_0) 2\pi i\delta(k^2 - m^2)\theta(k_0)) \end{aligned} \tag{4.42}$$

which is illustrated in Fig. 46 where the propagator with a cross stands for $2\pi i\delta(M^2 - p^2)\theta(p_0)$ or $2\pi i\delta(k^2 - m^2)\theta(k_0)$ for M and π-mesons, respectively.

In general, imaginary parts of diagrams are given by so-called "Cutkovsky rules": the discontinuity ($\equiv 2i\Im$) of a Feynman diagram is the sum of all possible cuts and on each cut

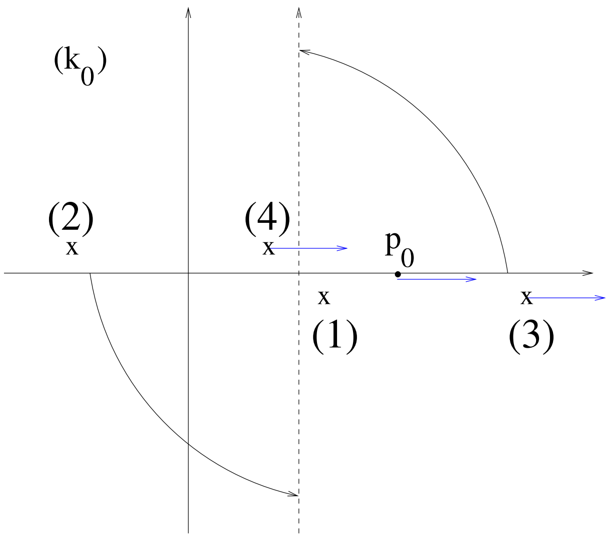


Figure 45. Singularities in complex k_0 plane for the self-energy integral at $p^2 > 0$.

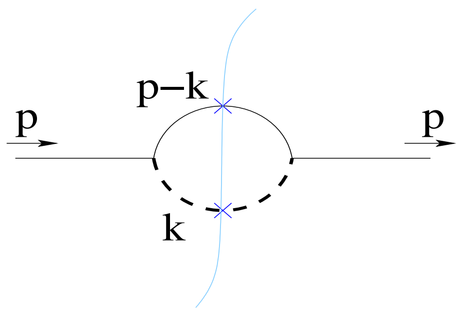


Figure 46. Imaginary part of the self-energy diagram

the corresponding propagator $(M^2 - p^2 - i\epsilon)^{-1}$ is replaced by $2\pi i\delta(M^2 - p^2)\theta(p_0)$ depicted by the corresponding line with a cross. In our example there is only one possible cut colored in blue in Fig. 46.

From Cutkovsky rules one immediately gets that vacuum bubbles have no imaginary parts since conservation of energy forbids creation of physical particles from the vacuum. Thus, the vacuum energy is real and the corresponding factor B in Eq. (4.29) is a real (albeit infinite) number so the factor e^{-iVTB} in Eq. (4.30) is indeed a pure phase shift which will be canceled in all physical cross sections.

Part X

4.5 Green functions and time evolution

To get the Feynman rules for the Green functions in the relativistic theory, we have postulated that, due to locality, all interactions can be reduced to elementary interaction process $M \Rightarrow M + \pi$ repeated it arbitrary number of times. Technically, it means that the Green functions for the (interacting) particles can be obtained by the convolution of the free Green functions with the amplitude of elementary interaction - vertex. But until we also specified the meaning of the Green functions in the relativistic theory we have actually postulated rules for construction of some artifact which may have nothing to do with real life. So, the next step (and actually it should be the first step) is to postulate that the meaning of the Green functions to be the same as in the non-relativistic case - namely, that Green fuctions describe the time evolution of our particles.

Let us recall the relation between time evolution and Green function in the non-relativistic theory of one particle interacting with time-independent potential. In order to study the time evolution, it is convenient to consider a finite-time scattering with the potential turned on $t = t_1$ and off at $t = t_2$, see Fig. (47).

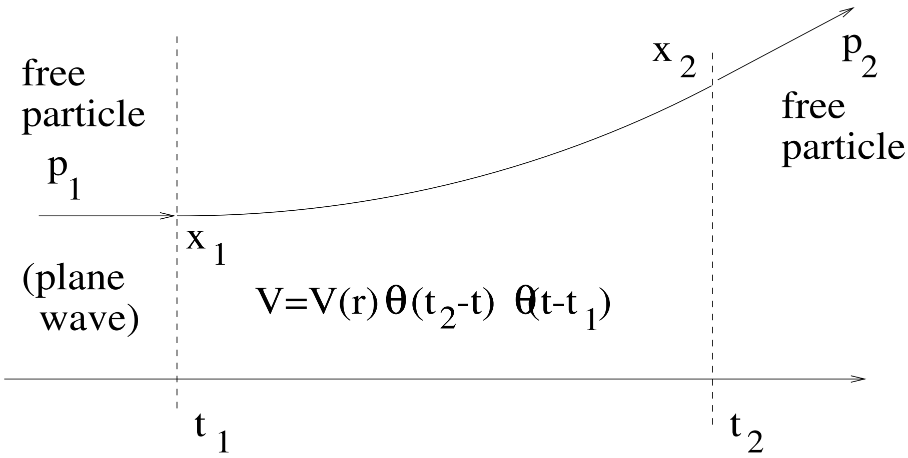


Figure 47. The finite-time scattering setup in the NR quantum mechanics

At time $t < t_1$ the particle moves as a plane wave (say, with momentum p_1):

$$t < t_1 : \qquad \Psi(x) = \Psi_{p_1}(x) = \frac{1}{L^{3/2}} e^{-i\frac{p_1^2}{2m} + i\vec{p}\vec{r}} \tag{4.43}$$

As usually, $x_1 \equiv (t_1, \vec{r}_1)$ etc. At time $t_2 > t > t_1$ the evolution of the particle is described by the Green function G^{NR} defined as a sum of Feynman diagrams (2.3.3) so

$$\Psi(\vec{r}_2, t_2) = \int d^3r_1 G^{NR}(\vec{r}_2, t_2; \vec{r}_1, t_1)\Psi_i(\vec{r}_1, t_1) \tag{4.44}$$

At time $t > t_2$ (after the scattering) the particle again moves as a superposition of the plane waves

$$t > t_2 : \qquad \Psi(x) = \int \frac{d^3p_2}{(2\pi)^3} U_{\vec{p}_2,\vec{p}_1}(t_2, t_1)\Psi_{p_2}(x) \tag{4.45}$$

where the “matrix elements of the evolution matrix" $U_{\vec{p}_2,\vec{p}_1}(t_2,t_1)$ are given by the overlap integrals of the wavefunction (4.44) with the plane waves:

$$U_{\vec{p}_2,\vec{p}_1}(t_2,t_1) = \int d^3r_2 d^3r_1 \Psi^*_{p_2}(x_2) G^{NR}(x_2;x_1)\Psi_{p_1}(x_1) \tag{4.46}$$

The evolution matrix is the S-matrix (2.5.4) for the time-truncated potential $V(r)\theta(t_2 - t)\theta(t-t_1)$. In terms of usual approach to quantum mechanics the evolution operator is

$$U(t_2,t_1) = e^{-iH(t_2-t_1)} \tag{4.47}$$

($H = H_0 + V(r)$ is the Hamiltionian) and

$$U_{\vec{p}_2,\vec{p}_1}(t_2,t_1) = \langle \vec{p}_2|e^{-iH(t_2-t_1)}|\vec{p}_1\rangle \tag{4.48}$$

where $\langle \vec{p}_2|$ and $|\vec{p}_1\rangle$ are Dirac bra and ket vectors for plane waves.

In the limit $t_1 \to -\infty, t_2 \to \infty$ our problem reduces to the scattering from the time-independent potential $V(r)$. (Actually, our finite-time setup with the subsequent limit $t_1 \to -\infty, t_2 \to \infty$ is a rigorous way to approach the scattering from the time-independent potential). It is easy to see that in this limit the matrix elements of the U-matrix reduce to the matrix elements of the S-matrix defined by eq. (2.5.4).

$$\lim_{t_2\to\infty}\lim_{t_1\to-\infty} U_{\vec{p}_2,\vec{p}_1}(t_2,t_1) = S(\vec{p}_2,\vec{p}_1) \tag{4.49}$$

In the relativistic theory we do not have the Schrödinger equation which defines the evolution operator accordoing to (4.48), but the definition (4.46) can be generalized to the relativistic situation. To this end, let us *postulate* that our Green functions obtained from the assumption of the locality of the interaction describe the time evolution of particle just as in the non-relativistic case. The crucial difference with the non-relativistic situation is that the number of particles is not conserved (and therefore our time evolution operator will be matrix in the space of states, see the discussion below).

In order to study the time evolution in relativistic quantum mechanics we will consider the same finite-time scattering setup as in the above NR case: we switch the interaction on at $t = t_1$ and turn it off at $t = t_2$, see Fig. (48).

Formally, we replace the coupling constant λ in our Feynman diagrams by the time-dependent coupling constant $\lambda(t) = \lambda\Theta(t-t_1)\Theta(t_2-t)$

$$\lambda \to \lambda\Theta(t-t_1)\Theta(t_2-t) \tag{4.50}$$

At this step we loose the relativistic invariance of our approach but it will be restored when we take the limit $t_2 \to \infty,\ t_1 \to -\infty$.

Let us for simplicity take the initial state at $t = t_1$ to be a free-particle plane wave with momentum p_1.

$$\tilde{\phi}_{p_1}(x_1) = \frac{1}{\sqrt{2E_1L^3}} e^{-ip_1x_1}\Big|_{p_{10}=E_1=\sqrt{\vec{p}_1^2+M^2}} \tag{4.51}$$

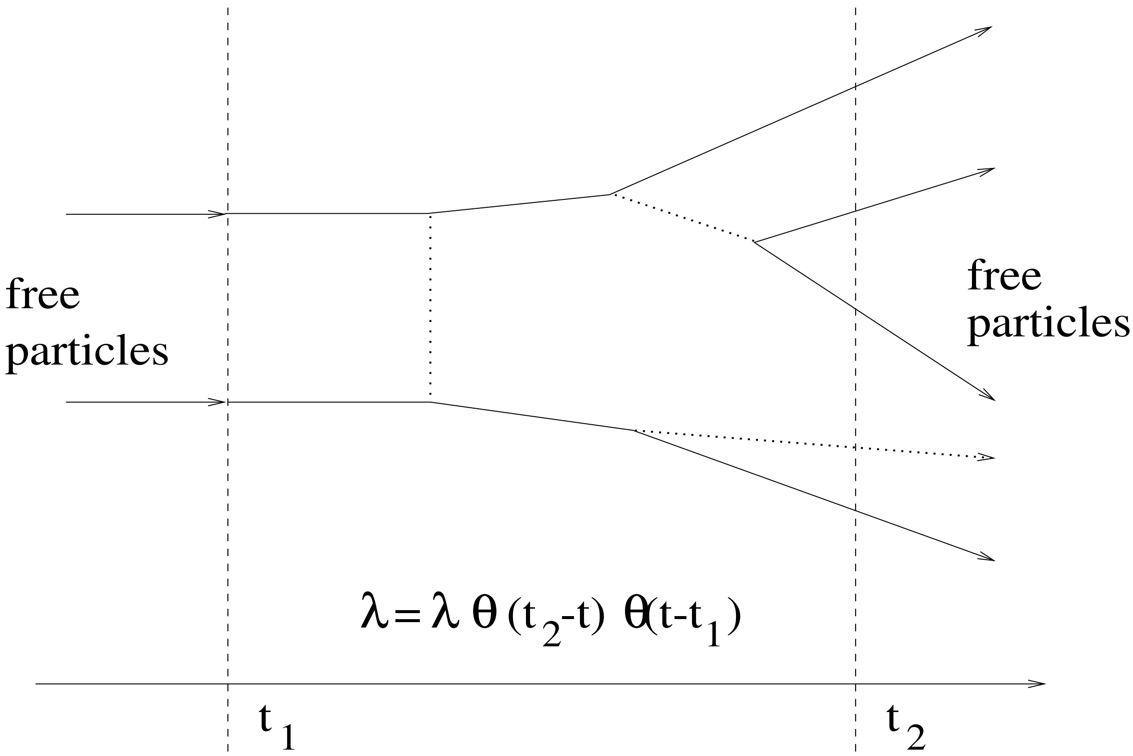


Figure 48. The finite-time scattering setup in relativistic theory

(the normalization is 1 particle in the large box with side L, cf. eq. (3.23) and eq. (3.25) so we put the label ˜ above the plane wave as in Lecture III):

$$\int d^3r \tilde{\phi}^*_{\vec{p}}(t,\vec{r}) i \overset{\leftrightarrow}{\frac{d}{dt}} \tilde{\phi}_{\vec{p}}(t,\vec{r}) = 1$$
$$\int d^3r \tilde{\phi}^*_{\vec{p}}(t,\vec{r}) i \overset{\leftrightarrow}{\frac{d}{dt}} \tilde{\phi}_{\vec{p}'}(t,\vec{r}) = 0, \qquad \vec{p} \neq \vec{p}' \tag{4.52}$$

As we discussed just above, in the trivial order in λ it simply moves freely without interaction so the wavefunction at the time $t = t_2$ is the same plane wave:

$$\tilde{\phi}_p(x_2) = \int d^3R_1 G_0(x_2,x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} \tilde{\phi}(x_1) = \frac{1}{\sqrt{2p_0L^3}} e^{-ipx_2}\Big|_{p_0=\sqrt{\vec{p}^2+M^2}} \tag{4.53}$$

(In this Section we denote the M-meson spatial coordinates by capital R's).

In the lowest nontrivial order in λ there is one elementary $M \to M\pi$ vertex in our disposal so we must have one M and one π-meson in the final state ($\equiv$ at time $t = t_2$), see Fig. 49. The wavefunction of this state at $t = t_2$ has the form (see Fig. 49):

$$\tilde{\phi}_{M\pi}(t_2,\vec{R}_2,\vec{r}_2) = \int d^3r_1 G(y_2,x_2;x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} \tilde{\phi}_p(x_1) \tag{4.54}$$

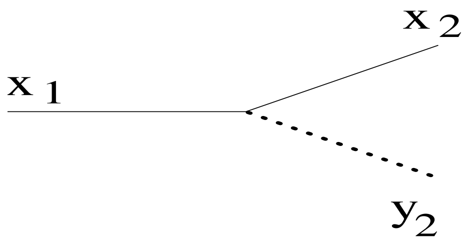


Figure 49. The $M \Rightarrow M + \pi$ transition

and the probability to find M-meson and the π-meson at time $t = t_2$ is:

$$P_{M\pi}(t_2) = \int d^3R_2 d^3r_2 \tilde{\phi}^*_{\pi M}(x_2, y_2))\; i\; \frac{\overleftrightarrow{d}}{dx_{20}}\; i\; \frac{\overleftrightarrow{d}}{dy_{20}}\; \tilde{\phi}_{\pi M}(x_2, y_2)\Bigg|_{x_{20}=y_{20}=t_2} \tag{4.55}$$

The status of the formula for the probability to find two (and more) particles is the educated guess which will be confirmed by the calculation of the probability conservation - just as in the case of one-particle probability density. The form of the expression in r.h.s of eq. (4.55) can be guessed from the consideration of the two non-interacting particles which can be described by the wavefunction

$$\phi_{M\pi}(x, y) = \phi_M(x)\phi_\pi(y) \tag{4.56}$$

(where $x = t, R$ and $y = t, r$) which is the product of two independent wavefunctions $\phi_M(x)$ and $\phi_(x)$. The probability density to find the M-meson in the point R and the π-meson in the point r (at time t) is then the product of the one-particle probability densities

$$\rho_{M\pi}(R, r) = \rho_M(R)\rho_\pi(r) \tag{4.57}$$

which are given by our usual expressions, see e.g. eq. (8.64). In terms of the two-particle wavefunction this probability density (4.57) can be written down as follows:

$$\rho_{M\pi}(R, r) = \phi^*_{M\pi}(x, y) i\; \frac{\overleftrightarrow{d}}{dx_0}\; i\; \frac{\overleftrightarrow{d}}{dy_0}\; \phi_{M\pi}(x, y)\Bigg|_{x_0=y_0=t} \tag{4.58}$$

and we simply assume that for the general situation of non-factorized $\phi_{M\pi}$ wave functions the probability density is given by the same formula. The justification of this assumption will be given below when we'll learn that the probability defined in such a way conserves. [21]

[21] Let us compare this to the non-relativistic approach where we have **(i)** Schrödinger equation , and **(ii)** interpretation of the square of the wavefunction as a probability density $\rho = |\Psi|^2$. *Both* of these statements follow from nowhere - they are guesses which are to be confirmed by the experiments. The same logic is true for the relativistic theory - only instead of the Schrödinger equation we have **(i)** a set equations of the (4.54) type describing the time evolution of the wavefunction of the initial state in terms of corresponding Green functions and instead of the simple formula $\rho = |\Psi|^2$ we have **(ii)** a set of formulas for probabilities of the eq. (4.55) type. But the meaning is the same - both **i** and **ii** are wild guesses to be confirmed by the experiment (and our arguments in favor of particular form for the expressions for the probability densities like (4.55) are, in fact, the checks of self-consistency of our formulas).

There is one more $O(\lambda)$ contribution to M-meson wave function at time t_2 given by the diagram shown in Fig. 50

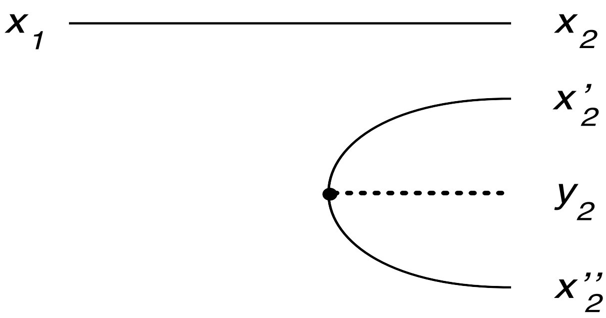


Figure 50. Four-particle component of the wavefunction at time t_2 in the λ^1 order

$$\begin{aligned}&\tilde{\phi}_{MMM\pi}(t_2,\vec{r}_2,\vec{r}'_2,\vec{R}) \\ &= \int d^3R_1\ G_{(1)}(x'_2,x''_2,y_2)G_0(x_2;x_1)i\ \overset{\leftrightarrow}{\frac{d}{dt_1}}\ \tilde{\phi}_p(x_1) \quad = \tilde{\phi}_p(x_2)G_{(1)}(x'_2,x''_2,y_2)\end{aligned} \tag{4.59}$$

The Green function $G_{(1)}(x_2,x'_2,y_2)$ describes the creation of two M-mesons and one π-meson from the vacuum (which can happen at short times due to Heisenberg uncertainty principle $\Delta E\Delta t\sim 1$). Hereafter he label $..._{(i)}$ denotes the order of perturbation theory.

In the next order in coupling constant our M-meson can emit extra π-meson or the π-meson can decouple into MM pair [22], so we have the three-particle component of the wavefunction in the form (see Fig. 51) The three-particle component of the wavefunction

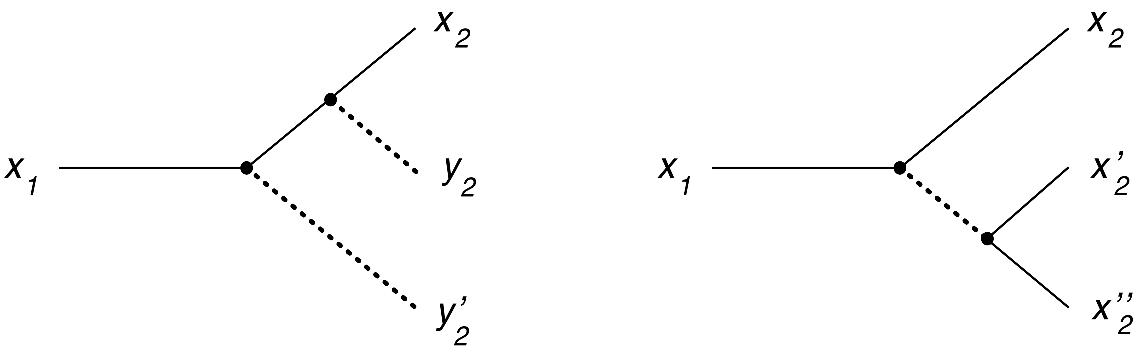


Figure 51. Three-particle component of the wavefunction at time t_2 in the λ^2 order

at time t_2 in the λ^2 order is a column with two elements corresponding to $M\pi\pi$ and MMM components of the wave function:

$$\begin{aligned}\tilde{\phi}_{M\pi\pi}(t_2,\vec{r}_2,\vec{r}'_2,\vec{R}) &= \int d^3R_1G_{(2)}(y_2,y'_2,x_2;x_1)i\ \overset{\leftrightarrow}{\frac{d}{dt_1}}\ \tilde{\phi}_p(x_1) \\ \tilde{\phi}_{MMM}(t_2,\vec{R}_2,\vec{R}'_2,\vec{R}''_2) &= \int d^3R_1G_{(2)}(x_2,x'_2,x''_2;x_1)i\ \overset{\leftrightarrow}{\frac{d}{dt_1}}\ \tilde{\phi}_p(x_1)\end{aligned} \tag{4.60}$$

[22]In addition, there is a bunch of diagrams with particles created from the vacuum of the Fig. 50 type

where $G_{(2)}(x_2, x_2', x_2"; x_1)$ and $G^{(2)}(y_2, y_2', x_2; x_1)$ are the Green functions for the $M \Rightarrow MMM$ and $M \Rightarrow M\pi\pi$ transitions, respectively. The corresponding probabilities has the form:

$$
\begin{aligned}
&P_{MMM}(t_2) = \\
&\frac{1}{3!}\int d^3R_2 d^3R_2' d^3R_2'' \tilde{\phi}^*_{MMM}(x_2, x_2', x_2'')\, i\, \frac{\overleftrightarrow{d}}{dx_{20}}\, i\, \frac{\overleftrightarrow{d}}{dx'_{20}}\, i\, \frac{\overleftrightarrow{d}}{dx''_{20}}\, \tilde{\phi}_{MMM}(x_2, x_2', x_2'')\Big|_{x_{20}=x'_{20}=x''_{20}=t_2} \\
&P_{M\pi\pi}(t_2) = \\
&\frac{1}{2!}\int d^3R_2 d^3r_2 d^3r_2' \tilde{\phi}^*_{M\pi\pi}(x_2, y_2, y_2'))\, i\, \frac{\overleftrightarrow{d}}{dx_{20}}\, i\, \frac{\overleftrightarrow{d}}{dy_{20}}\, i\, \frac{\overleftrightarrow{d}}{dy'_{20}}\, \tilde{\phi}_{M\pi\pi}(x_2, y_2, y_2')\Big|_{x_{20}=y_{20}=y'_{20}=t_2}
\end{aligned}
\tag{4.61}
$$

where the combinatorial factors $\frac{1}{n!}$ take into account the identity of the particles in the final state. Apart from that, in this order there is a correction to the one-particle wave function given by the diagrams in Fig. 52 which has the form:

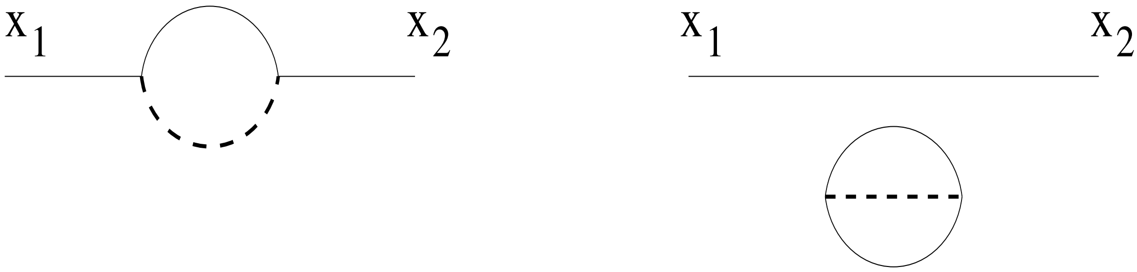


Figure 52. Corrections to the one-particle component of the wave function $\sim \lambda^2$

$$
\tilde{\phi}_M(t_2, R_2) = \int d^3R_1 G^{(2)}(x_2 - x_1) i\, \frac{\overleftrightarrow{d}}{dt_1}\, \tilde{\phi}_p(x_1) \;+\; \tilde{\phi}_p(x_2)(-iVTB) \tag{4.62}
$$

where the vacuum bubble B is defined in Eq. (4.29). (As we shall see below, vacuum bubbles do not contribute to probability conservation). The reason for this correction is as follows: since the M-meson can emit π-mesons the total probability to discover a single M-meson in the space at time t_2 should be less than 1 because there is a positive probability to create some other particles, e.g. π-mesons, and on the other hand, the total probability to have anything should be conserved.

Let us check the conservation of probability in the second order in λ. In this order the wavefunction of our state (which was a plane wave at $t = t_1$) is a four-row vector (in the so-called Fock space):

$$\tilde{\phi}_M(x_2) = (1 - iVTB)\tilde{\phi}_p(x_2) + \int d^3R_1 G_{(2)}(x_2 - x_1) i \overleftrightarrow{\frac{d}{dt_1}} \tilde{\phi}_p(x_1) \tag{4.63}$$

$$\tilde{\phi}_{M\pi}(x_2, y_2) = \int d^3R_1 G_{(1)}(y_2, x_2; x_1) i \overleftrightarrow{\frac{d}{dt_1}} \tilde{\phi}_p(x_1)$$

$$\tilde{\phi}_{MMM}(x_2, x_2', x"_2) = \int d^3R_1 G_{(2)}(x_2, x_2', x"_2; x_1) i \overleftrightarrow{\frac{d}{dt_1}} \tilde{\phi}_p(x_1)$$

$$\tilde{\phi}_{M\pi\pi}(x_2, y_2, y_2') = \int d^3R_1 G_{(2)}(y_2, y_2' x_2; x_1) i \overleftrightarrow{\frac{d}{dt_1}} \tilde{\phi}_p(x_1)$$

$$\phi_{MMM\pi}(x_2, x_2', x"_2, y_2) = G_{(1)}(x_2', x"_2, y_2) \int d^3R_1 G_0(x_2; x_1) i \overleftrightarrow{\frac{d}{dt_1}} \tilde{\phi}_p(x_1) \;=\; \tilde{\phi}_p(x_2) G_{(1)}(x_2', x"_2, y_2)$$

Thus, the probability to find one M-meson at time t_2 is (cf eq. 4.62):

$$P_M(t_2) = \int d^3R_2 \tilde{\phi}_M^*(t_2, R_2) i \overleftrightarrow{\frac{d}{dt_2}} \tilde{\phi}_M(t_2, R_2) \tag{4.64}$$

the probability to find M-meson and one π-meson is given by Eq. (4.55), the probability to find three M's and π-meson is

$$\begin{aligned} P_{MMM\pi}(t_2) \;&=\; \frac{1}{2!} \int d^3R_2 d^3R_2' d^3R_2'' d^3r_2 \\ &\times \tilde{\phi}^*_{MMM\pi}(x_2, x_2', x_2'', y_2)) \; i \overleftrightarrow{\frac{d}{dx_{20}}} \; i \overleftrightarrow{\frac{d}{dx_{20}'}} \; i \overleftrightarrow{\frac{d}{dx_{20}''}} \; i \overleftrightarrow{\frac{d}{dy_{20}}} \; \tilde{\phi}_{MMM\pi}(x_2.x_2', x_2'', y_2) \Big|_{x_{20}=x_{20}'=x_{20}''=y_{20}=t_2} \end{aligned} \tag{4.65}$$

and that is all, because the probability to find three particles is $\sim \lambda^4$ at best, see Eq. (4.61). So, if we sum all the probabilities in the order up to λ^2, we obtain:

$$\begin{aligned} P_M(t_2) + P_{M\pi}(t_2) + P_{MMM\pi}(t_2) \;&=\; 1 + (-iVTB) + iVTB \\ &+ \int d^3R_1 d^3R_2 \tilde{\phi}_p^*(x_2) i \overleftrightarrow{\frac{d}{dt_2}} G_{(2)}(x_2 - x_1) i \overleftrightarrow{\frac{d}{dt_1}} \tilde{\phi}_p(x_1) \\ &+ \int d^3R_1 d^3R_2 \tilde{\phi}_p^*(x_2) i \overleftrightarrow{\frac{d}{dt_1}} G^*_{(2)}(x_1 - x_2) i \overleftrightarrow{\frac{d}{dt_2}} \tilde{\phi}_p(x_2) \\ &+ \int d^3R_1 d^3R_1' d^3R_2 d^3r_2 \tilde{\phi}_p^*(x_1') i \overleftrightarrow{\frac{d}{dt_1'}} G^*_{(1)}(x_1'; x_2, y_2) \; i \overleftrightarrow{\frac{d}{dx_{20}}} \; i \overleftrightarrow{\frac{d}{dy_{20}}} \; G_{(1)}(x_2, y_2; x_1) i \overleftrightarrow{\frac{d}{dt_1}} \tilde{\phi}_p(x_1) \Big|_{x_{20}=y_{20}=t_2} \\ &+ \int d^3R_2 d^3R_2' d^3R"_2 d^3r_2 \phi^*_{p'}(x_2') G^*_{(1)}(x_2, x"_2, y_2) \; i \overleftrightarrow{\frac{d}{dx_{20}}} \; i \overleftrightarrow{\frac{d}{dx_{20}'}} \; i \overleftrightarrow{\frac{d}{dx"_{20}}} \; i \overleftrightarrow{\frac{d}{dy_{20}}} \; G_{(1)}(x_2', x"_2, y_2) \phi_p(x_2) \Big|_{x_{20}=x_{20}'=x"_{20}=y_{20}=t_2} \end{aligned} \tag{4.66}$$

First, one observes that the contribution of vacuum bubble cancels. As to the last four terms in the r.h.s. of this equation, I demonstrate in the Appendix F that the sum of these terms vanishes and therefore the probability is conserved. One immediate consequence of this fact is that the probability to discover the single M-boson once we switched on the interaction is less than 1.

Part XI

In higher orders in λ we can have as many particles in our final state as we wish, so the general form of the wavefunction at $t = t_2$ is an infinite column:

$$\begin{aligned}
\tilde{\phi}_M(x_2) &= \int d^3R_1 G(x_2 - x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} \tilde{\phi}_p(x_1) \\
\tilde{\phi}_{M\pi}(x_2, y_2) &= \int d^3R_1 G(y_2, x_2; x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} \tilde{\phi}_p(x_1) \\
\tilde{\phi}_{MMM}(x_2, x_2', x"_2) &= \int d^3R_1 G(x_2, x_2', x"_2; x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} \tilde{\phi}_p(x_1) \\
\tilde{\phi}_{M\pi\pi}(x_2, y_2, y_2') &= \int d^3r_1 G(y_2, y_2' x_2; x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} \tilde{\phi}_p(x_1) \\
\tilde{\phi}_{MMM\pi}(x_2, x_2', x_2'', y_2) &= \int d^3R_1 G(y_2, x_2, x_2', x_2''; x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} \tilde{\phi}_p(x_1) \\
&\dots.
\end{aligned} \tag{4.67}$$

where $G's$ are the exact Green functions. Now we shall project these states at $t = t_2$ into plane waves. Note that the projection will have extra $i \overset{\leftrightarrow}{\frac{d}{dt}}$ in comparison to ordinary Fourier transform (e.g. as in non-relativistic quantum mechanics) due to the different orthogonality condition for our plane waves (cf. eq. (3.27):

$$\int d^3r \; \phi^*_{\vec{p}}(t, \vec{r}) \; i \overset{\leftrightarrow}{\frac{d}{dt}} \; \phi_{\vec{p}'}(t, \vec{r}) = (2\pi)^3 \delta(\vec{p} - \vec{p}') \tag{4.68}$$

Here our plane waves are normalized as in the continuum spectrum and thus do not have tildes. It is convenient also to change at this point the normalization of the initial plane wave to (4.68) so we multiply both sides of equations like (4.67) by $L^{3/2}$ which effectively means wiping the label $\tilde{..}$ from both sides. Expanding the components of wavefunction into corresponding plane waves, one has:

$$\begin{aligned}
\phi_M(x_2) &= \int \frac{d^3p_2}{(2\pi)^3} \phi_{p_2}(x_2) U(t_2, t_1)_{p_2;p} \qquad (4.69) \\
\phi_{M\pi}(x_2, y_2) &= \int \frac{d^3p_2}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \phi_{p_2}(x_2) \phi_{k_2}(y_2) U(t_2, t_1)_{p_2, k_2;p} \\
\phi_{MMM}(x_2, x_2', x"_2) &= \int \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_2'}{(2\pi)^3} \frac{d^3p"_2}{(2\pi)^3} \phi_{p_2}(x_2) \phi_{p_2'}(x_2') \phi_{p"_2}(x"_2) U(t_2, t_1)_{p_2, p_2', p"_2;p}
\end{aligned}$$

and so on, where

$$
\begin{aligned}
U_{t_2,t_1}(p_2;p_1) &= \int d^3R_2 d^3R_1 \phi^*_{p_2}(x_2) i \overset{\leftrightarrow}{\frac{d}{dt_2}}\, G(x_2,x_1) i\, \overset{\leftrightarrow}{\frac{d}{dt_1}}\, \phi_{p_1}(x_1) = (2\pi)^3\delta(\vec{p}_2-\vec{p}_1)(1+O(\lambda^2)) \\
U_{t_2,t_1}(p_2,k_2;p_1) &= \int d^3R_2 d^3r_2 d^3R_1 \phi^*_{p_2}(x_2)\phi^*_{k_2}(y_2)\, i\, \overset{\leftrightarrow}{\frac{d}{dx_{20}}}\, i\, \overset{\leftrightarrow}{\frac{d}{dy_{20}}}\Bigg|_{x_{20}=y_{20}=t_2} G(x_2,y_2;x_1) i\, \overset{\leftrightarrow}{\frac{d}{dt_1}}\, \phi_{p_1}(x_1) \\
U_{t_2,t_1}(p_2,p_2',k_2;p_1) &= ...
\end{aligned} \tag{4.70}
$$

The elements of the "evolution matrix" $U_{t_2,t_1}(p_2,..p_2^{(m)},k_2...k_2^{(n)};p_1)$ give the amplitudes to observe our state (which was one-particle plane wave at $t=t_1$) at time $t=t_2$ as a set of m M-mesons and n π-mesons with momenta $p_2,p_2',...p_2^{(m)}$ and $k_2,k_2',...k_2^{(n)}$. The conservation of the probability (4.66) in terms of operator U has the simple form:

$$
\begin{aligned}
\textstyle\sum_{m,n} \frac{1}{m!n!} \int \frac{d^3p_2}{(2\pi)^3}....\frac{d^3k_2^{(n)}}{(2\pi)^3} U^*_{t_2,t_1}(p_2,..p_2^{(m)},k_2...k_2^{(n)};q_1)U_{t_2,t_1}(p_2,..p_2^{(m)},k_2...k_2^{(n)};p_1)& \\
= (2\pi)^3\delta(\vec{p}_1-\vec{q}_1)&
\end{aligned} \tag{4.71}
$$

where the r.h.s. is not 1 because we have multiplied our initial wavefunction by $L^{3/2}$ so the normalization for the free propagation without scattering is now (4.68) rather than (4.52).

Let us prove that the formulas (4.66) and (4.71) for the conservation of the probability are equivalent. For definiteness, consider the $P_{M\pi}$ term (the fourth line of Eq. (4.66)). The $M\pi$ contribution to Eq. (4.71) has the form

$$
\textstyle\int \frac{d^3p_2}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} U^*_{t_2,t_1}(p_2,,k_2;q_1)U_{t_2,t_1}(p_2,k_2;p_1) \tag{4.72}
$$

where $U_{t_2,t_1}(p_2,k_2;p_1)$ is defined in eq. (4.70). We will use the property that $\overset{\leftrightarrow}{\frac{d}{dx_{i0}}}$ in this definition may be replaced by $2\frac{d}{dx_{i0}}$ or $-2\ \overset{\leftarrow}{\frac{d}{dx_{i0}}}$. At first, we shall prove that when we integrate an arbitrary Green function with the plane wave at $t=t_2$ the direction of the arrow does not matter:

$$
\begin{aligned}
&\int d^3R_2 \phi_p(x_2) i\, \overset{\leftrightarrow}{\frac{d}{dx_{20}}}\Bigg|_{x_{20}=t_2} G(x_2,z_1,...z_n) = \\
&= 2\int d^3R_2 \phi_p(x_2) i \frac{d}{dx_{20}}\Bigg|_{x_{20}=t_2} G(x_2,z_1,...z_n) \\
&= -2\int d^3R_2 \phi_p(x_2) i\, \overset{\leftrightarrow}{\frac{d}{dx_{20}}}\Bigg|_{x_{20}=t_2} G(x_2,z_1,...z_n)
\end{aligned} \tag{4.73}
$$

where $z_1,...z_n$ may correspond to M or π meson tails. To demonstrate this, note that each Green function $G(x_2,z_1,...z_n)$ can be represented as shown in Fig. (53)

$$
G(x_2,z_1,...z_n) = \int dz\ G'(z,z_1,...z_n)N_0(x_2-z) \tag{4.74}
$$

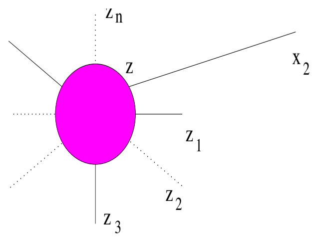


Figure 53.

where z is the position of the vertex where the line starting at x ends and $G'(x_1, ..., y_n, z)$ is the rest of the diagram. Also, since all the inetractions are restricted to the region $t < t_2$ (see the setup in Fig. 48) we can replace the Green function $N_0(x_2 - z)$ by the M-meson propagation function

$$L_0(x_2 - z) = \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-iE_p(t_2 - z_0) + i\vec{p}(\vec{R}_2 - \vec{z})} \tag{4.75}$$

(cf. 3.47). With this representation in mind, we must prove that

$$\begin{aligned} &\int d^3R_2\ \phi_p^*(x_2) i \frac{\overleftrightarrow{d}}{dx_{20}}\Big|_{x_0 = t} L_0(x_2 - z) = \\ &= 2 \int d^3R_2\ \phi_p^*(x_2) i \frac{d}{dx_{20}}\Big|_{x_{20} = t_2} L_0(x_2 - z) \\ &= -2 \int d^3R_2\ \phi_p^*(x_2) i \frac{\overleftarrow{d}}{dx_{20}}\Big|_{x_{20} = t_2} L_0(x_2 - z) \end{aligned} \tag{4.76}$$

which can be easily seen from the explicit form of the plane wave and L_0 (both right and left differentiations bring the same extra factor E_{p_2}).

Now we can return to the proof of eq. (4.72). We have seen that we can choose any direction of the arrow in $\frac{d}{dt}$ provided that we take into account the corresponding signs and 2's. (We have demonstrated it for the M-meson plane wave but the same proof can be repeated for the π-meson plane wave as well).

Let us choose the direction of the arrows as follows ($x'_1=(t_1,\vec{R'}_1),\ x_1=(t_1,\vec{R}_1)$):

$$\begin{aligned}
U_{t_2,t_1}(p_2,k_2;p_1) &= 4\int d^3\tilde{R}_2 d^3\tilde{r}_2 d^3R_1 \phi^*_{p_2}(\tilde{x}_2)\phi^*_{k_2}(\tilde{y}_2)\, i\frac{d}{d\tilde{x}_{20}} i\frac{d}{d\tilde{y}_{20}}\Big|_{\tilde{x}_{20}=\tilde{y}_{20}=t_2} G(\tilde{x}_2,\tilde{y}_2;\tilde{x}_1) i\, \frac{\overleftrightarrow{d}}{dt_1}\, \phi_{p_1}(x_1) \\
U^*_{t_2,t_1}(p_2,k_2;p_1) &= 4\int d^3R'_2 d^3r'_2 d^3R'_1 \phi^*_{p_1}(x'_1) i\, \frac{\overleftrightarrow{d}}{dt_1}\, G^*(x'_2,y'_2;x'_1)\left[i\frac{d}{dx'_{20}}\Big|_{x'_{20}=t_2}\phi_{p_2}(x'_2)\right]\left[i\frac{d}{dy'_{20}}\Big|_{y'_{20}=t_2}\phi_{k_2}(x'_2)\right]
\end{aligned} \tag{4.77}$$

If we substitute these equations into eq. (4.72) and use

$$\begin{aligned}
2\int\frac{d^3p_2}{(2\pi)^3}\left[i\frac{d}{dx'_{20}}\Big|_{x'_{20}=t_2}\phi_{p_2}(x'_2)\right]\phi^*_{p_2}(\tilde{x}_2) &= \delta(\vec{R}'_2-\vec{\tilde{R}}_2) \\
2\int\frac{d^3k_2}{(2\pi)^3}\left[i\frac{d}{dy'_{20}}\Big|_{y'_{20}=t_2}\phi_{k_2}(x'_2)\right]\phi^*_{k_2}(\tilde{y}_2) &= \delta(\vec{r}'_2-\vec{\tilde{r}}_2)
\end{aligned} \tag{4.78}$$

we get (after renaming integration variables $\tilde{R}_2\to R_2$ and $\tilde{r}_2\to r_2$)

$$\begin{aligned}
&\int\frac{d^3p_2}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}U^*_{t_2,t_1}(p_2,,k_2;q_1)U_{t_2,t_1}(p_2,k_2;p_1) \qquad (4.79) \\
&= 4\int d^3R'_1 d^3\tilde{R}_1 d^3R_2 d^3r_2 \phi^*_{p_1}(x'_1) i\, \frac{\overleftrightarrow{d}}{dt_1}\, G^*_{(1)}(x'_1;x_2,y_2)\, i\frac{d}{dx_{20}} i\frac{d}{dy_{20}} G_{(1)}(x_2,y_2;x_1) i\, \frac{\overleftrightarrow{d}}{dt_1}\, \phi_p(x_1)\Big|_{x_{20}=y_{20}=t_2}
\end{aligned}$$

In a way similar to the proof of eq. (4.73) we can demonstrate that the direction of the arrow in $\frac{d}{dx_{20}}$ or $\frac{d}{dy_{20}}$ does not matter. To this end we represent each of the Green functions $G_{(1)}(x_2,y_2;x_1)$ and $G^*_{(1)}(x'_1;x_2,y_2)$ in the form (4.74), use the fact that $x_{20}=t_2$ is larger than any time z associated with the interaction vertex which leads to replacement of $N_0(G_0)$ by L_0 (K_0), and use the equation

$$\int d^3R_2 N^*_0(z'-x_2)\frac{d}{dt}N_0(x_2-z) = -\int d^3R_2\left[\frac{d}{dt}N^*_0(z'-x_2)\right]N_0(x_2-z).$$

Thus, we have reproduced the fourth line in eq. (4.66). In a similar way one can prove that the formula (4.72) represent the conservation of probability for the terms in higher orders in perturbation theory. Note that the conservation of probabilty in the form (4.71) is more transparent than in the form (4.66) since it tells us that the sum of all probabilities to observe our initial state at $t=t_2$ as a superposition of the different plane waves is equal to 1 (in the finite box; in the conutinuum limit it is equal to the product of δ-functions).

A very important practical case is the two-particle scattering - the time evolution of the two-particle state. Let us consider the scattering of M-mesons as an example. At the time $t=t_1$ our initial state is a superposition of the two plane waves corresponding to the two incoming M-mesons: [23]

$$\phi_{MM}(x_1,x'_1)=\phi_{p_1}(x_1)\phi_{p'_1}(x'_1) \tag{4.80}$$

[23] Here again the alternative rigorous procedure is to switch on the interaction at $t=t_1$ and turn it off at $t=t_2$, then before $t=t_1$ we indeed have the superposition of the two free plane waves with momenta p_1 and p'_1 whose time slice at $t=t_1$ coincides with $\psi_{MM}(x_1,x'_1)$

As we discussed above, this state at $t = t_2$ can be described by the infinite column

$$\phi_\pi(y_2) \tag{4.81}$$
$$\phi_{MM}(x_2, x_2') \tag{4.82}$$
$$\phi_{MM\pi}(x_2, x_2', y_2) \tag{4.83}$$
$$\phi_{MM\pi\pi}(x_2, x_2', y_2, y_2') \tag{4.84}$$
$$\phi_{MMMM}(x_2, x_2', x_2'', x_2''') \tag{4.85}$$
$$.... \tag{4.86}$$

where , for example, the two-M two-π component of the $t = t_2$ wavefunction is

$$\phi_{MM\pi\pi}(x_2, x_2', y_2, y_2') = \int d^3R_1 d^3R_1' G(x_2, x_2', y_2, y_2'; x_1, x_1') i \frac{\overleftrightarrow{d}}{dx_{10}}\ i\ \frac{\overleftrightarrow{d}}{dx_{10}'}\ \psi_{MM}(x_1, x_1') \tag{4.87}$$

and the probability to find our $t = t_2$ state in the form of 2 M-mesons and two π-mesons is correspondingly

$$\rho_{MM\pi\pi}(t_2; r_2, r_2', R_2, R_2') = \tag{4.88}$$
$$\phi^*_{MM\pi\pi}(x_2, x_2', y_2, y_2') i \frac{\overleftrightarrow{d}}{dx_{20}}\ i \frac{\overleftrightarrow{d}}{dx_{20}'}\ i \frac{\overleftrightarrow{d}}{dy_{20}}\ i \frac{\overleftrightarrow{d}}{dy_{20}'}\Bigg|_{x_{20}=x_{20}'=y_{20}=y_{20}'=t_2} \phi_{MM\pi\pi}(x_2, x_2', y_2, y_2')$$

Again, let us project our final state on the plane waves:

$$\begin{aligned}
\phi_\pi(y_2) &= \int \frac{d^3k_2}{(2\pi)^3} \phi_{k_2}(y_2) U(t_2, t_1)_{k_2;p_1,p_1'} \\
\phi_{MM\pi}(x_2, x_2', y_2) &= \int \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_2'}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \phi_{p_2}(x_2) \phi_{p_2'}(x_2) \phi_{k_2}(y_2) U_{t_2,t_1}(p_2, p_2', k_2; p_1, p_1') \\
\phi_{MM\pi\pi}(x_2, x_2', y_2, y_2') &= \int \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_2'}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_2'}{(2\pi)^3} \phi_{p_2}(x_2) \phi_{p_2'}(x_2') \phi_{k_2}(y_2) \phi_{k_2'}(y_2') U_{t_2,t_1}(p_2, p_2', k_2, k_2'; p_1, p_1') \\
&...
\end{aligned} \tag{4.89}$$

where

$$\begin{aligned}
U_{t_2,t_1}(k_2; p_1, p_1') &= \int d^3r_2 d^3R_1 d^3R_1' \phi^*_{p_2}(y_2) i \frac{\overleftrightarrow{d}}{dt_2}\ G(y_2, x_1, x_1')\ i\ \frac{\overleftrightarrow{d}}{dx_{10}}\ i\ \frac{\overleftrightarrow{d}}{dx_{10}'}\Bigg|_{x_{10}=x_{10}'=t_1} \phi_{MM}(x_1, x_1') \\
U_{t_2,t_1}(p_2, p_2'; p_1, p_1') &= \int d^3R_2 d^3R_2' d^3R_1 d^3R_1' \phi^*_{p_2}(x_2) \phi^*_{p_2'}(x_2')\ i \frac{\overleftrightarrow{d}}{dx_{20}}\ i \frac{\overleftrightarrow{d}}{dx_{20}'}\Bigg|_{x_{20}=y_{20}=t_2} G(x_2, x_2'; x_1, x_1') \\
&\quad \times\ i \frac{\overleftrightarrow{d}}{dx_{10}}\ i \frac{\overleftrightarrow{d}}{dx_{10}'}\Bigg|_{x_{20}=y_{20}=t_2} \phi_{MM}(x_1, x_1') \\
U_{t_2,t_1}(p_2, p_2', k_2; p_1, p_1')\ &= ...
\end{aligned} \tag{4.90}$$

The elements of the "evolution matrix" $U_{t_2,t_1}(p_2, ..p_2^{(m)}, k_2...k_2^{(n)}; p_1, p_1')$ give the amplitudes to observe our (two-particle at time t_1) state at time $t = t_2$ as a set of m M-mesons and n π-mesons with momenta $p_2, p'_2, ...p_2^{(m)}$ and $k_2, k'_2, ...k_2^{(n)}$. For the future uses, let us also write down explicitly the conservation of probability for this case (cf. eq. (4.71)):

$$\sum_{m,n} \frac{1}{m!n!} \int \frac{d^3p_2}{(2\pi)^3} .... \frac{d^3k_2^{(n)}}{(2\pi)^3} U^*_{t_2,t_1}(p_2,..p_2^{(m)}, k_2...k_2^{(n)}; q_1, q'_1) U_{t_2,t_1}(p_2,..p_2^{(m)}, k_2...k_2^{(n)}; p_1, p'_1) =$$
$$= (2\pi)^3\delta(\vec{p}_1 - \vec{q}_1)(2\pi)^3\delta(\vec{p}'_1 - \vec{q}'_1) \tag{4.91}$$

Part XII

4.6 S-matrix and the transition matrix

Let us summarize what we achieved in the previous section. We have considered the finite-time scattering problem: switched the interaction on at time $t = t_1$ and turned it off at $t = t_2$. If before the scattering (at $t = t_1$) we had the free propagation of, say, two M-mesons with momenta p_1, p'_1 then after the scattering (at $t = t_2$) we may have any number of particles and the amplitude of such scattering is determined by the matrix element of the evolution matrix $U_{t_2,t_1}(p_2,..p_2^{(m)}, k_2...k_2^{(n)}; p_1, p'_1)$ (where $p_2,..p_2^{(m)}, k_2...k_2^{(n)}$ are the momenta of final particles). The conservation of the probability for such $2 \Rightarrow m+n$ scattering process is given by eq. (4.91).

Now let us take the limit $t_1 \to -\infty$, $t_2 \to \infty$ which correspond to the scattering experiments in our macroworld. Then the $2 \Rightarrow m + n$ process will look as follows. At the remote past (at $t_1 \to -\infty$) we had the two freely moving M-mesons with momenta p_1 and p'_1. After the scattering we can observe any number of particles (allowed by the energy conservation) with the probability determined by the corresponding S- matrix element

$$S(p_2,..p_2^{(m)}, k_2...k_2^{(n)}; p_1, p'_1) \stackrel{\text{def}}{\equiv} \lim_{t_1\to-\infty, t_2\to\infty} U_{t_2,t_1}(p_2,..p_2^{(m)}, k_2...k_2^{(n)}; p_1, p'_1) \tag{4.92}$$

It is easy to note that our definition of the S-matrix is a straightforward generalization of the similar definition (2.48) for the non-relativistic theory. These matrix elements were defined as the coefficient functions of the projection of the final state on the plane waves - see e.g. eq. (2.92). The only difference is that in the relativistic case our wavefunction is an infinite column of the many-particle components $\phi_{M..M\pi...\pi}$ representing different possible outcomes for our scattering process, so our S-matrix has a complicated matrix structure. But for each given element of the S-matrix the formula is very similar to eq. (2.48). For example, the explicit formula for the matrix element describing the $MM \Rightarrow MM$ transition is:

$$S(p_1, p'_1; \to p_2, p'_2) = \lim_{t_1\to-\infty,\ t_2\to\infty} \int d^3R_2 d^3R'_2 d^3R_1 d^3R'_1 \tag{4.93}$$
$$\phi^*_{p_2}(x_2)\phi^*_{p'_2}(x'_2)\ i\frac{\overleftrightarrow{d}}{dx_{20}}\ i\frac{\overleftrightarrow{d}}{dx'_{20}}\Big|_{x_{20}=y_{20}=t_2} G(x_2, x'_2; x_1, x'_1)\ i\frac{\overleftrightarrow{d}}{dx_{10}}\ i\frac{\overleftrightarrow{d}}{dx'_{10}}\Big|_{x_{10}=y_{10}=t_1} \phi_{p_1}(x_1)\phi_{p'_1}(x'_1)$$

One may recall that in the non-relativistic case the connection between the matrix elements of the S-matrix and the Green functions had a much simpler form in the momentum representation. This is also true for the relativistic theory. Let us derive such formula for our example - $MM \Rightarrow MM$ transition. The Green function $G(x_2, x_2'; x_1, x_1')$ can be represented as follows:

$$G(x_2, x_2'; x_1, x_1') = N_0(x_2' - x_1')N_0(x_2 - x_1) + \int dz_2 dz_2' dz_1 dz_1' N_0(x_2 - z_2)N_0(x_2' - z_2')G^{amp}(z_2, z_2'; z_1, z_1')N_0(z_1 - x_1)N_0(z_1' - x_1') \quad (4.94)$$

where the first term corresponds to the free propagation without scattering and the $G^{amp}(z_2, z_2'; z_1, z_1')$ is the Green function with amputated legs, see Fig. 54 [24]

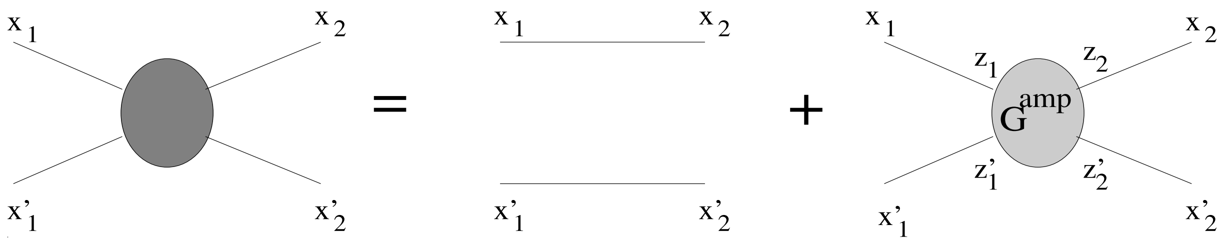


Figure 54. Total Green function for $MM \Rightarrow MM$ scattering as a sum of the disconnected and connected parts

Now, since $t_1 \to -\infty$, $t_2 \to \infty$ we can replace each of the Green functions N_0 in eq. (4.94) by the corresponding propagation function L_0 (cf. eq. 3.2.9)

$$L_0(t_2, \vec{r}_2, t_1, \vec{r}_1) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\sqrt{|\vec{p}|^2 + M^2}} e^{-i\sqrt{|\vec{p}|^2+M^2}(t_2-t_1)+i\vec{p}(\vec{r}_2-\vec{r}_1)} \quad (4.95)$$

so we obtain

$$G(x_2, x_2'; x_1, x_1') \overset{t_1\to-\infty,\ t_2\to\infty}{=} L_0(x_2 - x_1)L_0(x_2' - x_1') + \int dz_2 dz_2' dz_1 dz_1' L_0(x_2 - z_2)L_0(x_2' - z_2')G^{amp}(z_2, z_2'; z_1, z_1')L_0(z_1 - x_1)L_0(z_1' - x_1') \quad (4.96)$$

Now let us substitute this formula in our definition of the S-matrix element (4.93). Using the formulas

$$\int d^3R_2 \phi^*_{p_2}(x_2) i \frac{\overleftrightarrow{d}}{dx_{20}} L_0(x_2 - z_2) = \frac{1}{\sqrt{2E_2}} e^{ip_2 z_2}\Big|_{E_2 \equiv p_{20} = \sqrt{\vec{p}_2^2 + M^2}} = \phi^*_{p_2}(z_2)$$

$$\int d^3R_1 L_0(z_1 - x_1) i \frac{\overleftrightarrow{d}}{dx_{10}} \phi_{p_1}(x_1) = \frac{1}{\sqrt{2E_1}} e^{-ip_1 z_1}\Big|_{E_1 \equiv p_{10} = \sqrt{\vec{p}_1^2 + M^2}} = \phi_{p_1}(z_1) \quad (4.97)$$

[24] As we mentioned above, the non-connected diagrams of the Fig. 38 type bring no additional information so it is not nessesary to write them down. But here we want to keep such term for a while for completeness.

we can reduce the expression (4.93) to the Fourier transform of the amputated Green function times some simple factors:

$$S(p_2,p_2';p_1,p_1') = (2\pi)^6\delta(\vec{p}_2-\vec{p}_1)\delta(\vec{p}_2'-\vec{p}_1')+ \tag{4.98}$$
$$\int dz_1dz_2dz_1'dz_2' \, \frac{e^{ip_2z_2+ip_2'z_2'-ip_1z_1-ip_1'z_1'}}{\sqrt{2E_2}\sqrt{2E_2'}\sqrt{2E_1}\sqrt{2E_1'}}G^{\rm amp}(z_2,z_2';z_1,z_1')\Big|_{p_{20}=E_2,p_{20}'=E_2',p_{10}=E_1,p_{10}'=E_1'}$$
$$= \frac{1}{\sqrt{2E_2}}\frac{1}{\sqrt{2E_2'}}\frac{1}{\sqrt{2E_1}}\frac{1}{\sqrt{2E_1'}}G^{\rm amp}(p_2,p_2';p_1,p_1')\Big|_{p_{20}=E_2,p_{20}'=E_2',p_{10}=E_1,p_{10}'=E_1'}$$

where $E_i=\sqrt{\vec{p}_i^2+M^2}$.

In the next section we will demondtrate that this pattern is general: the element of S-matrix for the arbitrary $m+n\Rightarrow m'+n'$ transition is given by the amputated Green functions on the mass shell times factors $\frac{1}{\sqrt{2E_i}}$ for each particle.

As we have seen above, the (non-reduced) Green function in the momentum representation contains the δ - function corresponding to the conservation of the overall momentum. As in the case of thr non-relativistic scattering, it is convenient to introduce a transition matrix (T-matrix) with this δ-function excluded. We define:

$$S(p_2,p_2';p_1,p_2') = \tag{4.99}$$
$$(2\pi)^6\delta(\vec{p}_2-\vec{p}_1)\delta(\vec{p}_2'-\vec{p}_1')+(2\pi)^4\delta(p_2+p_2'-p_1-p_1')\frac{1}{\sqrt{2E_2}}\frac{1}{\sqrt{2E_2'}}\frac{1}{\sqrt{2E_1}}\frac{1}{\sqrt{2E_1'}}iT(p_2,p_2';p_1,p_1')$$

It is easy to see that the transition matrix is the amputated reduced Green function $\mathcal{G}(p_2,p_2';p_1,p_1')$ on the mass shell (see eq. 4.36):

$$T(p_2,p_2';p_1,p_1') = \lim_{p_1^2,p_1'^2,p_2^2,p_2'^2\to M^2}(p_1^2-M^2)(p_1'^2-M^2)(p_2^2-M^2)(p_2'^2-M^2)\mathcal{G}(p_2,p_2';p_1,p_1') \tag{4.100}$$

4.7 T-matrix and cross section for MM scattering

Let us now calculate the cross section of the $MM\Rightarrow MM$ scattering in the same way as we have done for the non-relativistic case (see Lecture III). Again, it is convenient to consider for a moment the plane waves $\tilde{\phi}_{\vec{p}}(x)$ (4.51) normalized to 1 particle per volume L^3, see eq. (4.52)). The probabilitry of the transition is proportional to the square of the matrix element of the S-matrix (since it is the S-matrix who has a meaning of the probability amplitude, see eq. (4.92)):

$$W_{fi} = \frac{[(2\pi)^4\delta(p_2+p_2'-p_1-p_1')]^2}{2E_22E_2'2E_1'2E_1}|\tilde{T}(p_2,p_2';p_1,p_1')|^2 \tag{4.101}$$

where

$$\tilde{T}(p_2,p_2';p_1,p_1') = \frac{1}{L^6}T(p_2,p_2';p_1,p_1') \tag{4.102}$$

As usual, label $\tilde{}$ means a T-matrix for the scattering of plane waves normalized according to (4.52). The eq. (4.101) is the probability of the transition (from the initial state of two

M-mesons with momenta p_1, p_1' to the final MM state with momenta p_2 and p_2') anywhere in the space and time. Therefore the rate of the transition is

$$\frac{1}{T}W_{fi} = \frac{[(2\pi)^4\delta(p_2+p_2'-p_1-p_1')]^2}{2E_2 2E_2' 2E_1' 2E_1 T}|\tilde{T}(p_2,p_2';p_1,p_1')|^2 \tag{4.103}$$

where $T = t_2 - t_1$ (and has nothing to do with T-matrix!). Since the number of final states in the momentum interval d^3p is $L^3\frac{d^3p}{(2\pi)^3}$ the rate of the probability for the two initial mesons to scatter in the interval of final states $d^3p_2 d^3p_2'$ is

$$\begin{aligned} dW_{fi} &= L^6\frac{d^3p_2}{(2\pi)^3}\frac{d^3p_2'}{(2\pi)^3}\frac{[(2\pi)^4\delta(p_2+p_2'-p_1-p_1')]^2}{2E_2 2E_2' 2E_1' 2E_1 T}|\tilde{T}(p_2,p_2';p_1,p_1')|^2 \\ &= \frac{1}{L^6 T}\frac{d^3p_2}{(2\pi)^3}\frac{d^3p_2'}{(2\pi)^3}\frac{[(2\pi)^4\delta(p_2+p_2'-p_1-p_1')]^2}{2E_2 2E_2' 2E_1' 2E_1}|T(p_2,p_2';p_1,p_1')|^2 \end{aligned} \tag{4.104}$$

The of square the δ-function can be unmasked using the same trick as in the non-relativistic case (see eq. (2.81):

$$\begin{aligned} &[(2\pi)^4\delta(p_2+p_2'-p_1-p_1')]^2 = \\ &(2\pi)^4\delta(p_2+p_2'-p_1-p_1')\int d^4x e^{-i(p_2+p_2'-p_1-p_1')x} = L^3T(2\pi)^4\delta(p_2+p_2'-p_1-p_1') \end{aligned} \tag{4.105}$$

The rate of probability (4.104) is reduced to:

$$dW_{fi} = \frac{1}{L^3}\frac{d^3p_2}{(2\pi)^3}\frac{d^3p_2'}{(2\pi)^3}\frac{(2\pi)^4\delta(p_2+p_2'-p_1-p_1')}{2E_2 2E_2' 2E_1' 2E_1}|T(p_2,p_2';p_1,p_1')|^2 \tag{4.106}$$

The last step is to divide this by flux and get the differential cross section. It is very simple to calculate the flux in the frame where the spatial components of the two colliding M-mesons are (anti)parallel. Then it is natural to define the flux as the number of particles crossing the unit of area (orthogonal to the momenta $\vec{p}_1 \parallel \vec{p}_1'$) from both sides, i.e. as the sum of the number of the particles crossing this area unit from left to right plus the number of particles crossing from right to left (see Fig. 55)

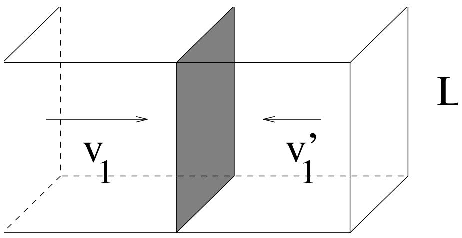


Figure 55. Flux for the two colliding beams

$$j = \frac{|\vec{v}_1|}{L^3} + \frac{|\vec{v}_1'|}{L^3} = \frac{1}{L^3}\left(\frac{|\vec{p}_1|}{E_1} + \frac{|\vec{p}_1'|}{E_1'}\right) \tag{4.107}$$

Note that for the particular choice of target frame (when $v_1' = 0$) this expression for the flux is $|\vec{v}_1|/L^3$ as you have seen in HW2. It is easy to see that the flux (4.107) can be rewritten in the form:

$$j = \frac{I}{4E_1E_1'L^3} \tag{4.108}$$

where

$$I \stackrel{\text{def}}{\equiv} 4\sqrt{(p_1 p_1')^2 - M^4} \tag{4.109}$$

is called the invariant flux [25]. Substituting these formulas for flux in the expression (4.106) we get:

$$d\sigma \stackrel{\text{def}}{\equiv} \frac{1}{j} dW_{fi} = \frac{1}{I} \frac{d^3 p_2}{(2\pi)^3 2E_2} \frac{d^3 p_2'}{(2\pi)^3 2E_2'} (2\pi)^4 \delta(p_2 + p_2' - p_1 - p_1') |T(p_2, p_2'; p_1, p_1')|^2 \tag{4.111}$$

It is easy to see that the cross section (4.112) is relativistic invariant since all the expresions in the r.h.s. are (recall that $\int \frac{d^3 p}{2p_0} = \int d^4 p \delta(p^2 - M^2) \Theta(p_0)$ is relativistic invariant). In order to get rid of the δ-function let us integrate the cross section over p_2'. The meaning of this procedure is that if we want to calculate the probability of one of the mesons to scatter in the momentum interval $d^3 p_2$ without paying attention to what the second meson is doing we must sum ($\equiv$ integrate) over all the possible momenta for the second meson. However, due to the conservation law the momentum of the second meson is fixed so the integration over the momentum of the second meson is trivial - this integration and $\delta^{(3)}(\vec{p}_2' + \vec{p}_2 - \vec{p}_1 + \vec{p}_1')$ simply cancel each other.

We obtain:

$$d\sigma = \frac{1}{I} \frac{d^3 p_2}{(2\pi)^3 4E_2 E_2'} 2\pi \delta(E_2 + E_2' - E_1 - E_1') |T(p_2, p_2'; p_1, p_1')|^2 \tag{4.112}$$

It is convenient to proceed further in the spherical coordinates in the center of mass frame. In this frame

$$E_1 = E_1', |\vec{p}_1| = |\vec{p}_1'|, E_2 = E_2', |\vec{p}_2| = |\vec{p}_2'| \tag{4.113}$$

so

$$I = 8E_1 |\vec{p}_1| \tag{4.114}$$

and also

$$d^3 p_2 = p_2^2 dp_2 d\Omega \tag{4.115}$$

where Ω is the spherical angle in our c.m. frame. Then our expression for the cross section (4.112) reduces to

$$d\sigma = \frac{1}{8E_1 |\vec{p}_1|} \frac{p_2^2 dp_2 d\Omega}{(2\pi)^3 4E_2^2} 2\pi \delta(2E_2 - 2E_1) |T(p_2, p_2'; p_1, p_1')|^2 \tag{4.116}$$

To get rid of the remaining δ-function let us integrate this expression over the final energy E_2. The meaning of this integration is as follows: since we are not interested in the energy of final particle, we must sum over all the possible energies (but due to the conservation

[25] If the masses of two colliding particles are different (say M and M') the eq. (4.107) is still true (it just counts the number of particles) but the invariant flux takes the form

$$I \stackrel{\text{def}}{\equiv} 4\sqrt{(p_1 p_1')^2 - M^2 M'^2} \tag{4.110}$$

where p is the momentum of the particle with mass M and p' with mass M'.

of energy E_2 is actually fixed). The final expression for the differential cross section of the scattering in the element of spherical angle Ω in the c.m. frame has the form:

$$\frac{d\sigma}{d\Omega} = \frac{|T(p_2, p_2'; p_1, p_1')|^2}{256 E_1^2 \pi^2} \tag{4.117}$$

The last step is to plug in the formula for the T-matrix (in the Born approximatiuon). First, the reduced Green function is given by eq. (4.22), see Fig. 24. Amputating the external legs we get the following expression for the T-matrix (see eq. (4.100)):

$$T(p_2, p_2'; p_1, p_1') = \frac{\lambda^2}{m^2 - t - i\epsilon} + \frac{\lambda^2}{m^2 - u - i\epsilon} + \frac{\lambda^2}{m^2 - s - i\epsilon} \tag{4.118}$$

where

$$\begin{aligned} s &= (p_1 + p_1')^2 \\ t &= (p_1 - p_2)^2 \\ u &= (p_1 - p_2')^2 \end{aligned} \tag{4.119}$$

are the so-called Mandelstam variables for the $2 \Rightarrow 2$ particle scattering. These variables are not independent: one may check that

$$s + t + u = 4M^2 \tag{4.120}$$

so only two of them are independent. The three terms in r.h.s. of eq. (4.118) correspond to the contributions of the diagrams in Fig. 24 a,b, and c, respectively. In the c.m. frame Mandelstam invariants take the form:

$$s = 4E_1^2, \quad t = -4|\vec{p}_1|^2 \sin^2\left(\frac{\theta}{2}\right), \quad u = -4|\vec{p}_1|^2 \cos^2\left(\frac{\theta}{2}\right) \tag{4.121}$$

where, as usually, θ is the angle between the initial momentum $\vec{p}_1$ and the final momentum $\vec{p}_2$. So, the final answer for the differential cross section is:

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^4}{256 E_1^2 \pi^2} \left(\frac{1}{m^2 + 4|\vec{p}_1|^2 \sin^2(\frac{\theta}{2})} + \frac{1}{m^2 + 4|\vec{p}_1|^2 \cos^2(\frac{\theta}{2})} - \frac{1}{4E_1^2 - m^2} \right)^2 \tag{4.122}$$

The total cross section is obtained by integration of eq. (4.122) over the spherical angle:

$$\sigma_{tot} = \frac{1}{2} \int d\Omega \frac{\lambda^4}{256 E_1^2 \pi^2} \left(\frac{1}{m^2 + 4|\vec{p}_1|^2 \sin^2(\frac{\theta}{2})} + \frac{1}{m^2 + 4|\vec{p}_1|^2 \cos^2(\frac{\theta}{2})} - \frac{1}{4E_1^2 - m^2} \right)^2 \tag{4.123}$$

where the factor $\frac{1}{2}$ is due to the fact that the two M-mesons in the final state are identical [26]. Performing the integration over $d\Omega = 2\pi \sin\theta d\theta$, one obtains:

$$\begin{aligned} &\sigma_{tot} \\ &= \frac{\lambda^4}{64\pi s} \left[\left(\frac{1}{|\vec{p}_1|^2 (2|\vec{p}_1|^2 + m^2)} - \frac{2}{|\vec{p}_1|^2 (s - m^2)} \right) \ln(1 + \frac{4|\vec{p}_1|^2}{m^2}) + \frac{4}{m^2 (m^2 + 4|\vec{p}_1|^2)} + \frac{2}{(s - m^2)^2} \right] \end{aligned} \tag{4.124}$$

[26] If you move your detector around the whole sphere you'll register each event twice. In other words, if you want to register each event only once, you should move your detector only over , say, upper semishpere, since the recoil M-meson will fly into the lower semishpere.

where $|\vec{p}_1|^2 = \frac{s}{4} - M^2$.

Let us compare our answer with the non-relativistic limit which should correspond to the MM scattering in the Yukawa potential. Suppose that $M \gg m$ and suppose that the momenta of the incoming (and outgoing) M-mesons are small in comparison to M (but not necessarily to m). Then

$$\vec{p}_1 = M\vec{v}_1, \quad \vec{p}_2 = M\vec{v}_2, \quad E_1 = M \tag{4.125}$$

It is easy to see that in the non-relativistic limit the last term in r.h.s. of eq. (4.122) is small in comparison to the first two ones, so we obtain:

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^4}{256M^2\pi^2}\left(\frac{1}{m^2 + 4|\vec{p}_1|^2\sin^2(\frac{\theta}{2})} + \frac{1}{m^2 + 4|\vec{p}_1|^2\cos^2(\frac{\theta}{2})}\right)^2 \tag{4.126}$$

Now let us recall the non-relativistic result for the scattering from Yukawa potential (HW1). If the particle with mass μ and momentum $\vec{p}_1$ is scattered from the Yukawa potential (2.86), the differential cross section has the form:

$$\frac{d\sigma}{d\Omega} = \left(\frac{2\mu V_0}{\alpha^2 + 4|\vec{p}_1|^2\sin^2(\frac{\theta}{2})}\right)^2 \tag{4.127}$$

The cross section for the scattering of the two identical particles with mass M interacting by (Yukawa) potential in the c.m. frame can be reduced to one-particle scattering with the effective mass $\mu = \frac{M}{2}$ so the differential cross section for the MM scattering with the Yukawa potential interaction is:

$$\frac{d\sigma}{d\Omega} = \left(\frac{MV_0}{\alpha^2 + 4|\vec{p}_1|^2\sin^2(\frac{\theta}{2})} + \frac{MV_0}{\alpha^2 + 4|\vec{p}_1|^2\cos^2(\frac{\theta}{2})}\right)^2 \tag{4.128}$$

where we have added the exchange term corresponsing to the case of the scattering in the angle $180^o - \theta$ which is undistinquishable from the scattering in the angle θ so we must add the amplitudes rather than the probabilities (cross sections). Now we see that the two expressions (4.126) and (4.128) are actually identical if one makes the identification:

$$\alpha = m, \quad V_0 = \frac{\lambda^2}{16\pi M^2} \tag{4.129}$$

The first of these properties tells us that the range of the Yukawa potential $\frac{1}{\alpha}$ is determined by the inverse π-meson mass, and indeed 1fm$\simeq (140MeV)^{-1}$. The second of these properties enable us to relate the coupling constant λ to the "experimentally observed" Yukawa constant V_0. In this artificial πM model it has not much sense, but in the case of electrodynamics of π-mesons this is the way how we will relate the π-meson-photon vertex to the experimentally obseved electric charge.

Homework assignment 3.

Suppose we have two sorts of M-mesons with masses M_1 and M_2 which interact with π-mesons with the same coupling constant λ so both the $M_1M_1\pi$ and $M_2M_2\pi$ vertices are equal to λ. Find the differential and total cross sections of $M_1M_2 \Rightarrow M_1M_2$ scattering in the first nontrivial order of perturbation theory (in the c.m. frame).

4.8 T-matrix and cross section in general case

Let us consider the general process of the πM transition shown in Fig. 56. We have the m_1 M-mesons with momenta $p_1, p'_1, ..p_1^{(m_1)}$ and n_1 π-mesons with momenta $k_1, k'_1, ...k_1^{(n_1)}$ in the initial state and m_2 M-mesons with momenta $p_2, p'_2, ..p_2^{(m_2)}$ and n_1 π-mesons with momenta $k_2, k'_2, ...k_2^{(n_2)}$ in the final state.

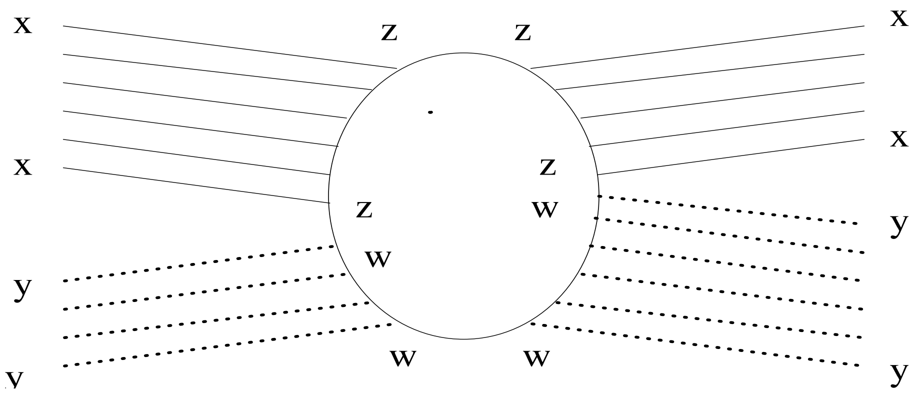


Figure 56. $M^{m_1}\pi^{n_1} \Rightarrow M^{m_2}\pi^{n_2}$ scattering

Using our definition of S-matrix (4.92) and the expressing the U-matrix in terms of the Green functions according to (4.90) we obtain (cf. eq. (4.93):

$$S(p_2, ...p_2^{(m_2)}, k_2, ...k_2^{(n_2)}; p_1, ...p_1^{(m_1)}, k_1, ...k_1^{(n_1)}) = \tag{4.130}$$

$$\lim_{t_1\to-\infty,\ t_2\to\infty} \int \Pi_{j_2} d^3R_2^{j_2} \Pi_{l_2} d^3r_2^{l_2} \Pi_{j_1} d^3R_1^{j_1} \Pi_{l_1} d^3r_1^{l_1} \phi^*_{p_2^{j_2}}(x_2^{j_2})\ i\ \overleftrightarrow{\frac{d}{dx_{20}^{j_2}}}\Big|_{x_{20}=t_2} \phi^*_{k_2^{l_2}}(y_2^{l_2})\ i\ \overleftrightarrow{\frac{d}{dy_{20}^{l_2}}}\Big|_{x_{20}=t_2}$$

$$G(x_2, ...x_2^{m_2}, y_2, ...y_2^{n_2}; x_1, ...x_1^{m_1}, y_1, ...y_1^{n_1})\ i\ \overleftrightarrow{\frac{d}{dx_{10}}}\Big|_{x_{10}=t_1} \phi_{p_1^{j_1}}(x_1^{j_1}) i\ \overleftrightarrow{\frac{d}{dy_{10}}}\Big|_{x_{10}=t_1} \phi_{k_1^{l_1}}(y_1^{l_1})$$

As we discussed above, at $t_1 \to -\infty, t_2 \to \infty$ one can write down the corresponding Green functions in r.h.s.of eq. (4.130) in the form (cf. eq. (4.94):

$$\begin{aligned} &G^{\text{connected}}(x_2, ...x_2^{m_2}, y_2, ...y_2^{n_2}; x_1, ...x_1^{m_1}, y_1, ...y_1^{n_1}) = \\ &\int \Pi_{j_2} dz_2^{j_2} \Pi_{l_2} dw_2^{l_2} \Pi_{j_1} dz_2^{j_1} \Pi_{l_1} dw_2^{l_1} L_0(x_2 - z_2)...L_0(x_2^{(m_2)} - z_2^{(m_2)}) K_0(y_2 - w_2)...K_0(y_2^{(n_2)} - w_2^{(n_2)}) \\ &\qquad G^{amp}(z_2, ...z_2^{m_2}, w_2, ...w_2^{n_2}; z_1, ...z_1^{m_1}, w_1, ...w_1^{n_1}) \\ &\qquad L_0(z_1 - x_1)...L_0(z_1^{(m_1)} - x_1^{(m_1)}) K_0(w_1 - y_1)...K_0(w_1^{(n_1)} - y_1^{(n_1)}) \end{aligned} \tag{4.131}$$

Using now the formulas

$$\int d^3R_2\phi^*_{p_2}(x_2)i\frac{\overleftrightarrow{d}}{dx_{20}}\,L_0(x_2-z_2) = \frac{1}{\sqrt{2E_2}}e^{ip_2z_2}\Big|_{E_2\equiv p_{20}=\sqrt{\vec{p}_2^2+M^2}} = \phi^*_{p_2}(z_2)$$
$$\int d^3r_2\phi^*_{k_2}(y_2)i\frac{\overleftrightarrow{d}}{dy_{20}}\,K_0(y_2-w_2) = \frac{1}{\sqrt{2E_2}}e^{ik_2w_2}\Big|_{E_2\equiv k_{20}=\sqrt{\vec{k}_2^2+m^2}} = \phi^*_{k_2}(w_2)$$
$$\int d^3R_1L_0(z_1-x_1)i\frac{\overleftrightarrow{d}}{dx_{10}}\,\phi_{p_1}(x_1) = \frac{1}{\sqrt{2E_1}}e^{-ip_1z_1}\Big|_{E_1\equiv p_{10}=\sqrt{\vec{p}_1^2+M^2}} = \phi_{p_1}(z_1)$$
$$\int d^3r_1K_0(w_1-y_1)i\frac{\overleftrightarrow{d}}{dy_{10}}\,\phi_{k_1}(y_1) = \frac{1}{\sqrt{2E_1}}e^{-ik_1w_1}\Big|_{E_1\equiv k_{10}=\sqrt{\vec{k}_1^2+m^2}} = \phi_{k_1}(w_1) \quad (4.132)$$

it is easy to obtain the general element of S-matrix in the form:

$$S^{\text{connected}}(p_2,...p_2^{(m_2)},k_2,...k_2^{(n_2)};p_1,...p_1^{(m_1)},k_1,...k_1^{(n_1)}) = \\ \Pi_{j_2}\frac{1}{\sqrt{2E_2^{j_2}}}\Pi_{l_2}\frac{1}{\sqrt{2E_2^{l_2}}}\Pi_{j_1}\frac{1}{\sqrt{2E_1^{j_1}}}\Pi_{l_1}\frac{1}{\sqrt{2E_1^{l_1}}} \\ G^{amp}(p_2,...p_2^{(m_2)},k_2,...k_2^{(n_2)};p_1,...p_1^{(m_1)},k_1,...k_1^{(n_1)})\Big|_{E^{(j)}=\sqrt{(\vec{p}^{(j)})^2+M^2},E^{(l)}=\sqrt{(\vec{k}^{(l)})^2+m^2}} \quad (4.133)$$

So, the matrix element of S-matrix for the arbitrary $m_1+n_1 \Rightarrow m_2+n_2$ transition is given by the amputated Green functions on the mass shell times factors $\frac{1}{2E_i}$ for each particle. The definition of the transition matrix is:

$$S^{\text{connected}}(p_2,...p_2^{(m_2)},k_2,...k_2^{(n_2)};p_1,...p_1^{(m_1)},k_1,...k_1^{(n_1)}) = \\ (2\pi)^4\delta(\textstyle\sum_{j_2}p_2^{(j_2)}+\sum_{l_2}k_2^{(l_2)}-\sum_{j_1}p_1^{(j_1)}-\sum_{l_1}k_1^{(l_1)})\Pi_{j_2}\frac{1}{\sqrt{2E_2^{j_2}}}\Pi_{l_2}\frac{1}{\sqrt{2E_2^{l_2}}}\Pi_{j_1}\frac{1}{\sqrt{2E_1^{j_1}}}\Pi_{l_1}\frac{1}{\sqrt{2E_1^{l_1}}} \\ iT(p_2,...p_2^{(m_2)},k_2,...k_2^{(n_2)};p_1,...p_1^{(m_1)},k_1,...k_1^{(n_1)}) \quad (4.134)$$

If we recall now the connection (4.36) between the Green function G and the reduced Green function $\mathcal{G}$ we see that the transition matrix is the amputated reduced Green function $\mathcal{G}(p_2,...p_2^{(m_2)},k_2,...k_2^{(n_2)};p_1,...p_1^{(m_1)},k_1,...k_1^{(n_1)})$ on the mass shell [27]:

$$T(p_2,...p_2^{(m_2)},k_2,...k_2^{(n_2)};p_1,...p_1^{(m_1)},k_1,...k_1^{(n_1)}) = \\ \lim_{(p^{(j)})^2\to M^2,\ (k^{(l)})^2\to m^2}\Pi_{j_2}[(p_2^{(j_2)})^2-M^2]\Pi_{l_2}[(k_2^{(l_2)})^2-m^2] \\ \Pi_{j_1}[(p_1^{(j_1)})^2-M^2]\Pi_{l_1}[(k_1^{(l_1)})^2-m^2]\mathcal{G}(p_2,...p_2^{(m_2)},k_2,...k_2^{(n_2)};p_1,...p_1^{(m_1)},k_1,...k_1^{(n_1)}) \quad (4.135)$$

Eq. (4.135) is the desired general expression for the transition matrix. Together with Feynman rules **I-V** for the reduced Green functions it gives a complete set of prescriptions for calculating of the probability of any transition in our model.

[27] Note that $G^{\text{amp}} = i(2\pi)^4\delta(\sum p_2+\sum k_2-\sum p_1-\sum k_1)(-1)^{m_1+n_1+m_2+n_2}\mathcal{G}^{\text{amp}}$ because when we amputate G we remove $\frac{1}{i(m^2-p^2-i\epsilon)}$ with each leg, whereas when we amputate $\mathcal{G}$ we remove $\frac{1}{(m^2-p^2-i\epsilon)}$. The extra i for each leg accounts for the difference between this formula and eq. (4.36).

Part XIII

4.9 Cross section for $2 \Rightarrow$ *many* particles scattering and optical theorem

From the practical point of view, a typical scattering process observed in the accelerator experiments has two particles in the initial state and arbitrary number of particles in the final state (as many as the initial energy permits). Let us write down the differential cross section for such $2 \Rightarrow m+n$ transition shown in Fig. 57 (for definiteness, we take 2 M-mesons in the initial state)

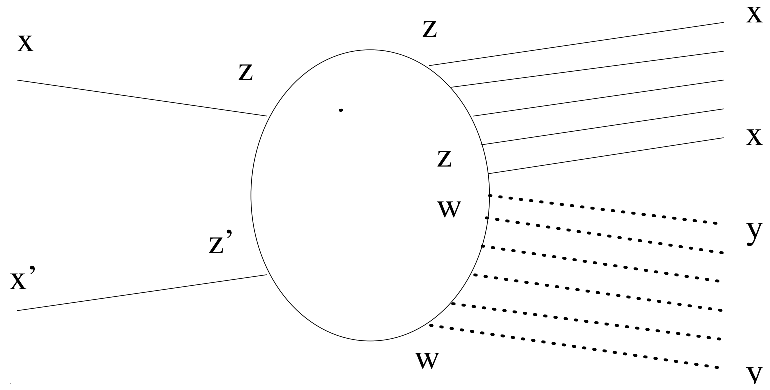


Figure 57. $MM \Rightarrow M^m\pi^n$ scattering

The relation between S and T-matrices (4.134) takes the form:

$$S^{\text{connected}}(p_2,...p_2^{(m_2)},k_2,...k_2^{(n_2)};p_1,p_1') = (2\pi)^4\delta(\textstyle\sum_{j_2} p_2^{(j_2)} + \sum_{l_2} k_2^{(l_2)} - p_1 - p_1')\frac{1}{\sqrt{2E_1}}\frac{1}{\sqrt{2E_1'}}\Pi_{j_2}\frac{1}{\sqrt{2E_2^{(j_2)}}}\Pi_{l_2}\frac{1}{\sqrt{2E_2^{(l_2)}}} \; iT(p_2,...p_2^{(m_2)},k_2,...k_2^{(n_2)};p_1,p_1') \qquad (4.136)$$

Similarly to the case of $MM \Rightarrow MM$ scattering, the probability rate of the transition from the state with momenta p_1, p_1' to the final states where the momentum of the first M-meson lies in the interval d^3p_2 around the momentum $\vec{p}_2$, the second in the interval d^3p_2' around the momentum $\vec{p}_2'$,... and the momentum of n_2th π-meson in the interval $d^3k_2^{(n_2)}$ around $\vec{k}_2^{(n_2)}$ has the form:

$$\begin{aligned}\frac{1}{T}dW_{fi} &= \Pi_{j_2}L^3\frac{d^3p_2^{j_2}}{(2\pi)^3 2E_2^{(j_2)}}\Pi_{l_2}L^3\frac{d^3k_2^{l_2}}{(2\pi)^3 2E_2^{(l_2)}}\frac{[(2\pi)^4\delta(\sum_{j_2} p_2^{(j_2)} + \sum_{l_2} k_2^{(l_2)} - p_1 - p_1')]^2}{2E_1' 2E_1 T} \\ &\quad |\tilde{T}(p_2,...p_2^{(m_2)},k_2,...k_2^{(n_2)};p_1,p_1')|^2 \\ &= \frac{1}{L^6 T}\Pi_{j_2}\frac{d^3p_2^{j_2}}{(2\pi)^3 2E_2^{(j_2)}}\Pi_{l_2}\frac{d^3k_2^{l_2}}{(2\pi)^3 2E_2^{(l_2)}}\frac{[(2\pi)^4\delta(\sum_{j_2} p_2^{(j_2)} + \sum_{l_2} k_2^{(l_2)} - p_1 - p_1')]^2}{2E_1' 2E_1} \\ &\quad |T(p_2,...p_2^{(m_2)},k_2,...k_2^{(n_2)};p_1,p_1')|^2 \end{aligned} \qquad (4.137)$$

Here again $T = t_2 - t_1$ and we have used the connection

$$\tilde{T}(p_2,...p_2^{(m_2)},k_2,...k_2^{(n_2)};p_1,p_1') = \frac{1}{L^3}\frac{1}{L^{\frac{3}{2}(m+n)}}T(p_2,...p_2^{(m_2)},k_2,...k_2^{(n_2)};p_1,p_1') \qquad (4.138)$$

between T-matrix calculated for the plane waves normalized by the condition of one particle per volume L^3 and the continuum-spectrum condition (3.27). Making again use of the formula

$$[(2\pi)^4\delta(\sum_{j_2} p_2^{(j_2)} + \sum_{l_2} k_2^{(l_2)} - p_1 - p_1')]^2 = L^3 T(2\pi)^4\delta(\sum_{j_2} p_2^{(j_2)} + \sum_{l_2} k_2^{(l_2)} - p_1 - p_1') \quad (4.139)$$

(cf. eq. (4.105)) we easily obtain the time rate of the probability in the form:

$$\frac{1}{T}dW_{fi} = \quad (4.140)$$
$$\frac{L^{-3}}{4E_1E_1'}(2\pi)^4\delta(\sum_{j_2} p_2^{(j_2)} + \sum_{l_2} k_2^{(l_2)} - p_1 - p_1')|T(p_2,...p_2^{(m_2)},k_2,...k_2^{(n_2)};p_1,p_1')|^2\Pi_{j_2}\frac{d^3p_2^{j_2}}{(2\pi)^3 2E_2^{(j_2)}}\Pi_{l_2}\frac{d^3k_2^{l_2}}{(2\pi)^3 2E_2^{(l_2)}}$$

The last step is to divide this rate by the initial M-meson flux $j = \frac{I}{4E_1E_1'L^3}$ (see eqs. (4.108),(4.109)). Thus, the final answer for the differential cross section of $MM \Rightarrow M^{m_2}\pi^{n_2}$ scattering has the form:

$$d\sigma = \frac{1}{j}\frac{dW_{fi}}{T} = \quad (4.141)$$
$$= \frac{1}{I}\Pi_{j_2}\frac{d^3p_2^{j_2}}{(2\pi)^3 2E_2^{(j_2)}}\Pi_{l_2}\frac{d^3k_2^{l_2}}{(2\pi)^3 2E_2^{(l_2)}}(2\pi)^4\delta(\sum_{j_2} p_2^{(j_2)} + \sum_{l_2} k_2^{(l_2)} - p_1 - p_1')|T(p_2,...p_2^{(m_2)},k_2,...k_2^{(n_2)};p_1,p_1')|^2$$

Note that this expression is relativistic invariant.

The total cross section is given by integration of Eq. (4.141) over momenta of final particles

$$\begin{aligned}\sigma_{\rm tot}(p_1,p_1' \to \text{any}) &= \\ &= \frac{1}{I}\sum_{m,n}\frac{1}{m!n!}\int\Pi_j\frac{d^3p_2^{(j)}}{(2\pi)^3 2E_2^{(j)}}\Pi_l\frac{d^3k_2^{(l)}}{(2\pi)^3 2E_2^{(l)}}(2\pi)^4\delta(\sum_{j_2} p_2^{(j_2)} + \sum_{l_2} k_2^{(l_2)} - p_1 - p_1') \\ &\qquad T^*(p_2,..p_2^{(m)},k_2...k_2^{(n)};p_1,p_1')T(p_2,..p_2^{(m)},k_2...k_2^{(n)};p_1,p_1') \qquad (4.142)\end{aligned}$$

The combinatorial factor $\frac{1}{m!n!}$ reflects the fact that if you integrate over all the momenta you'll count each event $m!n!$ times because mesons of the same kind are indistinguishable (see the discussion after Eq. (4.123)).

Let us derive now the optical theorem reflecting the property of conservation of probability (unitarity of S-matrix). The conservation of the probability for the transition of the two M-meson state at time $t = t_1$ into everything possible at the time $t = t_2$ is given by eq. (4.91):

$$\begin{aligned}&\sum_{m,n}\frac{1}{m!n!}\int\Pi_j\frac{d^3p_2^{(j)}}{(2\pi)^3}\Pi_l\frac{d^3k_2^{(l)}}{(2\pi)^3}U^*_{t_2,t_1}(p_2,..p_2^{(m)},k_2...k_2^{(n)};q_1,q_1')U_{t_2,t_1}(p_2,..p_2^{(m)},k_2...k_2^{(n)};p_1,p_1') = \\ &\qquad = (2\pi)^3\delta(\vec{p}_1 - \vec{q}_1)(2\pi)^3\delta(\vec{p}_1' - \vec{q}_1') \qquad (4.143)\end{aligned}$$

Taking the limit $t_2 \to \infty,\ t_1 \to -\infty$ we obtain:

$$\begin{aligned}&\sum_{m,n}\frac{1}{m!n!}\int\Pi_j\frac{d^3p_2^{(j)}}{(2\pi)^3}\Pi_l\frac{d^3k_2^{(l)}}{(2\pi)^3}S^*(p_2,..p_2^{(m)},k_2...k_2^{(n)};q_1,q_1')S(p_2,..p_2^{(m)},k_2...k_2^{(n)};p_1,p_1') = \\ &\qquad = (2\pi)^3\delta(\vec{p}_1 - \vec{q}_1)(2\pi)^3\delta(\vec{p}_1' - \vec{q}_1') \qquad (4.144)\end{aligned}$$

In the matrix notations it reads $S^\dagger S = 1$.

Now, let us separate the connected part of the S-matrix according to eq. (4.136):

$$S(p_2, ...p_2^{(m)}, k_2, ...k_2^{(n)}; p_1, p_1') =$$
$$\delta_{m2}\delta_{n0}[(2\pi)^3\delta(\vec{p}_1 - \vec{p}_2)(2\pi)^3\delta(\vec{p}_1' - \vec{p}_2') + (2\pi)^3\delta(\vec{p}_1 - \vec{p}_2')(2\pi)^3\delta(\vec{p}_1' - \vec{p}_2)$$
$$+(2\pi)^4\delta(\textstyle\sum_{j_2} p_2^{(j_2)} + \sum_{l_2} k_2^{(l_2)} - p_1 - p_1')\frac{1}{\sqrt{2E_1}}\frac{1}{\sqrt{2E_1'}}\Pi_{j_2}\frac{1}{\sqrt{2E_2^{j_2}}}\Pi_{l_2}\frac{1}{\sqrt{2E_2^{l_2}}}$$
$$iT(p_2, ...p_2^{(m_2)}, k_2, ...k_2^{(n_2)}; p_1, p_1') \tag{4.145}$$

Substituting the r.h.s. of this eq. into eq. (4.144) we obtain:

$$iT^*(q_1, q_1'; p_1, p_1') - iT(p_1, p_1'; q_1, q_1') =$$
$$\textstyle\sum_{m,n}\frac{1}{m!n!}\int \Pi_j \frac{d^3p_2^{(j)}}{(2\pi)^3 2E_2^{(j)}}\Pi_l \frac{d^3k_2^{(l)}}{(2\pi)^3 2E_2^{(l)}}(2\pi)^4\delta(\sum_{j_2} p_2^{(j_2)} + \sum_{l_2} k_2^{(l_2)} - p_1 - p_1')$$
$$T^*(p_2, ..p_2^{(m)}, k_2...k_2^{(n)}; q_1, q_1')T(p_2, ..p_2^{(m)}, k_2...k_2^{(n)}; p_1, p_1') \tag{4.146}$$

In order to get the total cross section, let us take the forward-scattering case: $q_1 = p_1$, $q_1' = p_1'$. It is easy to see then that the r.h.s. of equation (4.146) is the total $MM \Rightarrow$*everything* cross section up to the invariant flux (4.109) so we finally obtain:

$$\Im T(p_1, p_1'; p_1, p_1') = \frac{I}{2}\sigma_{tot} \tag{4.147}$$

This is the celebrated optical theorem: the total cross section is the imaginary part of the forward scattering amplitude divided by invariant flux of incoming particles [28].

5 Electrodynamics of scalar particles

Part XIV

5.1 Charged π-mesons and complex Klein-Gordon field

The neutral π-meson π^0 is the example of neutral scalar particles which are described by the real scalar field. In Nature here exist also the charged scalar particles π^+ and π^- with mass $m_{\pi^+} = m_{\pi^-} = m = 140MeV$ which can be described by the *complex* scalar field $\phi(x)$ satisfying the Klein-Gordon equation $(\Box + m^2)\phi(x) = 0$. As we discussed in Sect. 2C, the general solution of thee Klein-Gordon equation can be decomposed into the plane waves. Similarly to the case of real scalar field we can write down

$$\phi(x) = \phi_+^{\pi+}(x) + \phi_+^{\pi-*}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(e^{-itE_p + i\vec{p}\vec{r}}C(\vec{p}) + e^{itE_p - i\vec{p}\vec{r}}D^*(\vec{p})\right) \tag{5.1}$$

[28] Note that the optical theorem does not depend on the fact whether the coupling constant is small or large and it was indeed checked experimentally for the real strong-interacting π-mesons. So, experimental test of the optical theorem is one of the verifications of our rules for of calculating the probabilities of transitions and the whole underlying approach originating from the hypothesis of the locality of the interactions.

where, as usual, $E_p = \sqrt{|\vec{p}|^2 + m^2}$ [29] Note that unlike eq. (3.8), the positive and negative parts of the field in r.h.s. of eq. (5.1) are not complex conjugate since we no longer have the condition $\phi = \phi^*$. Instead, $\phi^*(x)$ is an independent field with the expansion into plane waves which is c.c. of eq. (5.1):

$$\phi^*(x) = \phi_+^{\pi^-}(x) + \phi_+^{\pi^+ *}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(e^{-itE_p + i\vec{p}\vec{r}} D(\vec{p}) + e^{itE_p - i\vec{p}\vec{r}} C^*(\vec{p}) \right) \tag{5.2}$$

In the case of complex field, we have not one but two quantities to which the probability meaning can be ascribed. They are:

$$\begin{aligned} \rho_{\pi^+}(x) &= \phi_+^{\pi^+ *}(x) i \overset{\leftrightarrow}{\frac{d}{dt}} \phi_+^{\pi^+}(x) \\ \rho_{\pi^-}(x) &= \phi_+^{\pi^- *}(x) i \overset{\leftrightarrow}{\frac{d}{dt}} \phi_+^{\pi^-}(x) \end{aligned} \tag{5.3}$$

and it is easy to check using the Klein-Gordon equation that they are conserved

$$\begin{aligned} \frac{d}{dt}\rho_{\pi^+}(t,\vec{R}) &= i\nabla_j[\phi_+^{\pi^+ *}(t,\vec{R})\nabla_j\phi_+^{\pi^+}(t,\vec{R}) - \phi_+^{\pi^+}(t,\vec{R})\nabla_j\phi_+^{\pi^+ *}(t,\vec{R})] \\ \frac{d}{dt}\rho_{\pi^-}(t,R) &= i\nabla_j[\phi_+^{\pi^- *}(t,\vec{R})\nabla_j\phi_+^{\pi^-}(t,\vec{R}) - \phi_+^{\pi^-}(t,\vec{R})\nabla_j\phi_+^{\pi^- *}(t,\vec{R})] \end{aligned} \tag{5.4}$$

Thus, for the complex scalar field we have two conserved integrals, $\int d^3r \rho_+(t,\vec{r})$ and $\int d^3r \rho_-(t,\vec{r})$, rather than one as in the case of real scalar field. Taking into consideration also that for the stationary state $\phi_+(x) = e^{-i\omega t}\phi(\vec{r})$ both of the quantities (5.3) are positive everywhere

$$\begin{aligned} \phi_+^{\pi^+}(x) = e^{-i\omega t}\phi_+^{\pi^+}(\vec{R}) &\rightarrow \rho_{\pi^+}(x) = 2\omega|\phi_+^{\pi^+}(R)|^2 \\ \phi_-^{\pi^-}(x) = e^{-i\omega t}\phi_-^{\pi^-}(\vec{R}) &\rightarrow \rho_{\pi^-}(x) = 2\omega|\phi_-^{\pi^-}(R)|^2, \end{aligned} \tag{5.5}$$

we conclude that for the free complex field we can construct two quantities (5.3) which have the meaning of local probability density. As we shall see below, they will correspond to the density of the π^+ and π^- particles.

In order to know that this complex scalar field corresponds to charged π-mesons we must study the interactions of these particles with the electromagnetic field [30]. As a first step, let us recall (and construct where necessary) the free Green functions in our theory. The Green function of the charged π-meson is the same as for the π^0 case considered in Sect. 3C with a (slightly different) mass corresponding to charged π-mesons. This is due to the fact that the wave function of the massive scalar particle with definite momentum $\vec{p}$ has the same form (3.26) for the scalar particles of all sorts since they obey the same Klein-Gordon equation. So we can simply repeat the steps which lead us from the eq. (3.26) to propagation function (3.38) and finally to Green function(3.50) and obtain the same result:

$$\mathcal{G}_0(p) = \frac{1}{m^2 - p^2 - i\epsilon} \tag{5.6}$$

[29] We have denoted the second term as $D^*(\vec{p})$ rather than $D(\vec{p})$ in view of the symmetry between eq. (5.1) and the complex conjugate equation (5.2).

[30] Before we have done that, we can claim instead that complex scalar field correspond, say, to the particles with charm - there is no way to identify charge in the non-interacting theory

where m is now the mass of charged π-mesons [31].

5.2 Quantum mechanics of photons

In the framework of classical relativistic mechanics the electromagnetic field is described by the field strength tensor $F^{\mu\nu}(x)$:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & H_z & -H_y \\ -E_y & -H_z & 0 & H_x \\ -E_z & H_y & -H_x & 0 \end{pmatrix} \tag{5.7}$$

where E and H are electric and magnetic fields and $x = (t, \vec{r})$ is a four-vector of coordinate. Classical dynamics of elecromagnetic field is determined by Maxwell equations:

$$\partial^\nu F_{\mu\nu}(x) = j_\mu(x) \tag{5.8}$$

$$\partial^\lambda F_{\mu\nu} + \partial^\nu F_{\lambda\mu} + \partial^\mu F_{\nu\lambda} = 0 \tag{5.9}$$

The relativistic invariance of Maxwell equations written down in terms of field tensor F is transparent.

It is convenient to introduce potentials $A_\mu(x)$ such as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.10}$$

then the equation (5.9) is trivially satisfied. However, the definition of the potentials according to (5.10) is ambiguous: one can always add the total derivative of an arbitrary scalar function $\Lambda(x)$:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \tag{5.11}$$

It is easy to see that the redefinition (5.11) of the potential does not change electric and magnetic fields. It is convenient to fix the potentials by additional Lorentz condition:

$$\partial_\mu A^\mu = 0 \tag{5.12}$$

then they satisfy the equations:

$$\Box A_\mu(x) = j_\mu(x), \qquad \Box \stackrel{\text{def}}{\equiv} \partial_\alpha \partial^\alpha = \frac{d}{dt}^2 - \frac{d}{dx_i}\frac{d}{dx_i} \tag{5.13}$$

For the free electromagnetic field the equation (5.13) reduces to

$$\Box A_\mu(x) = 0 \tag{5.14}$$

The combination of the two equations (5.14) and (5.12) is equivalent to the set of Maxwell equations (5.8),(5.9).

[31] As we shall show below, the fact that the masses of positively and negatively charged π-mesons are equal is a consequence of charge conservation. Anticipating this, we actually included this fact in the definition of the charged π-meson field from the beginning - we postulated that π^+ and π^- are described *by the same* scalar field.

Mathematically, the eq. (5.14) is the combination of four d'Alembert equations (one for each component μ)with the additional constraint (5.12).

The d'Alembert equation

$$\Box f(x) = 0 \tag{5.15}$$

is a Klein-Gordon equation (3.1) with $m = 0$ so its general solution has the form (cf. eq. (3.8)

$$f(x) = \int \frac{d^3k}{(2\pi)^3} (a(\vec{k})e^{-ikx} + a^*(\vec{k})e^{ikx})\Big|_{k_0=E_k=|\vec{k}|} \tag{5.16}$$

where $a(\vec{k})$ is an arbitrary function of 3-momentum $\vec{k}$.

For the electromagnetic field one gets

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \Big(\frac{a_\mu(\vec{k})}{\sqrt{2E_k}} e^{-ikx} + \frac{a^*_\mu(\vec{k})}{\sqrt{2E_k}} e^{ikx}\Big)\Big|_{k_0=E_k=|\vec{k}|} \tag{5.17}$$

where the functions $a_\mu(\vec{k})$ satisfy the restriction

$$k_\mu a^\mu(\vec{k}) = 0 \tag{5.18}$$

following from Lorentz condition

$$\partial^\mu A_\mu(x) = i \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} k^\mu(-a_\mu(\vec{k})e^{-ikx} + a^*_\mu(\vec{k})e^{ikx})\Big|_{k_0=E_k=|\vec{k}|} = 0 \tag{5.19}$$

There are four functions $a^\mu(\vec{k})$ and one restriction (5.18) which leaves three independent functions $a^\lambda(\vec{k})$ $(\lambda = 1, 2, 3)$. It is convenient to choose them in the form

$$a_\mu(\vec{k}) = \sum_{\lambda=1,2,3} e^\lambda_\mu(\vec{k}) a^\lambda(\vec{k}) \tag{5.20}$$

where $e^\lambda_\mu(\vec{k})$ are three vectors satisfying the condition

$$k^\mu e^\lambda_\mu(\vec{k}) = 0 \tag{5.21}$$

They are called the *polarization vectors* of the photon. The conventional choice for the polarization vectors is

$$e^{(1)}_\mu = (0, 1, 0, 0) \tag{5.22}$$

$$e^{(2)}_\mu = (0, 0, 1, 0) \tag{5.23}$$

$$e^{(3)}_\mu = k_\mu \tag{5.24}$$

in the frame where the Z axis is parallel to $\vec{k}$ (so that $k_\mu = (k, 0, 0, k)$). Substituting the eq. (5.20) in eq. (5.17) we obtain the expansion of the free electromagnetic field in the form which satisfies the Lorentz condition automatically

$$A_\mu(x) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3} \frac{e^\lambda_\mu(\vec{k})}{\sqrt{2E_k}} \left[a^\lambda(\vec{k})e^{-ikx} + a^{*\lambda}(\vec{k})e^{ikx}\right]\Big|_{k_0=E_k} \tag{5.25}$$

which satisfies the Lorentz condition automatically.

Next, we demonstrate that the "longitudinal" term in (5.25) proportional to $e^3_\mu(\vec{k}) = k_\mu$ can be gauged away by a suitable gauge transformation of the vector potentials $A_\mu(x)$. Let us demonstrate this explicitly. We write down the expansion (5.25) separating the contributions of the transverse and longitudinal photons:

$$
\begin{aligned}
A_\mu(x) &= \sum_{\lambda=1,2} \int \frac{d\!\!{}^-\!{}^3k}{\sqrt{2E_k}} e^\lambda_\mu(\vec{k}) \left[a^\lambda(\vec{k}) e^{-ikx} + a^{*\lambda}(\vec{k}) e^{ikx} \right] \\
&+ i \frac{\partial}{\partial x^\mu} \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} \left[a^{(3)}(\vec{k}) e^{-ikx} - a^{*(3)}(\vec{k}) e^{ikx} \right] \\
&= A^{tr}_\mu + \partial_\mu \Lambda(x)
\end{aligned}
\tag{5.26}
$$

We see that the difference between the total field A_μ and the physical transverse part A^{tr}_μ is a gauge transformation. This gauge transformation is allowed even after imposing the Lorentz condition (5.12). Indeed,

$$
\partial^\mu \left(\partial_\mu \Lambda(x) \right) \equiv \Box \Lambda(x) = 0 \tag{5.27}
$$

so we can perform the gauge transformation $A_\mu(x) \to A_\mu(x) - \partial_\mu \Lambda(x) = A^{\rm tr}_\mu(x)$ because it does not interfere with Lorentz condition due to eq. (5.27)

Performing this transformation, we get the electromagnetic field describing only physical transversely polarized degrees of freeedom:

$$
A_\mu(x) = \sum_{\lambda=1,2} \int \frac{d\!\!{}^-\!{}^3k}{\sqrt{2E_k}} e^\lambda_\mu(\vec{k}) \Big(a^\lambda(\vec{k}) e^{-ikx} + a^{*\lambda}(\vec{k}) e^{ikx} \Big) \Big|_{k_0=E_k} \tag{5.28}
$$

Similarly to the case of scalar particles, the positive-frequency part of this field

$$
f_\mu(x) = \sum_{\lambda=1,2} \int \frac{d\!\!{}^-\!{}^3k}{\sqrt{2E_k}} e^\lambda_\mu(\vec{k}) a^\lambda(\vec{k}) e^{-ikx} \tag{5.29}
$$

can serve as a wavefunction of the superposition of the free-photon states. The wavefunction of a single photon with momentum $\vec{k}$ and polarization λ ($= 1$ or 2) has the form

$$
(f^\lambda_k)_\mu(x) = \frac{e^\lambda_\mu(\vec{k})}{\sqrt{2|\vec{k}|}} e^{-i|\vec{k}|t + i\vec{k}\vec{r}} \tag{5.30}
$$

The wave functions (5.30) are normalized as follows:

$$
\int d^3r (f^\lambda_{\vec{k}})^*_\mu(x) i \overset{\leftrightarrow}{\frac{d}{dt}} (f^{\lambda'}_{\vec{k}'})^\mu(x) = e^\lambda_\mu(\vec{k}) e^{\lambda'\mu}(\vec{k}) (2\pi)^3 \delta(\vec{k} - \vec{k}') = -\delta_{\lambda\lambda'} (2\pi)^3 \delta(\vec{k} - \vec{k}') \tag{5.31}
$$

As usual, if we want to normalize the wave function (5.30) by the condition to have one photon in a large box with side L we should multiply it by the factor $\frac{1}{L^{3/2}}$:

$$
(\tilde{f}^\lambda_{\vec{k}})_\mu(x) = \frac{e^\lambda_\mu(\vec{k})}{\sqrt{2|\vec{k}|L^3}} e^{-ikx} \tag{5.32}
$$

The density of the probability is defined as [32]

$$\rho(x) = -(\tilde{f}^{\lambda}_{\vec{k}})^*_{\mu}(x) i \overset{\leftrightarrow}{\frac{d}{dt}} (\tilde{f}^{\lambda}_{\vec{k}})^{\mu}(x) \tag{5.33}$$

(where $\lambda = 1$ or 2) and it is easy to check that

$$\begin{aligned} &\rho(t, \vec{r}) = \tfrac{1}{L^3} \Rightarrow \\ &\int d^3 r \rho(t, \vec{r}) = 1 \end{aligned} \tag{5.34}$$

so indeed we have one photon in the space. As usually, in practical calculations of Feynman diagrams the continuous-spectrum normalization (5.31) is more convenient. We will use the one-photon normalization (5.32) only when we will relate Feynman amplitudes to the cross sections, as in the scalar case.

Part XV

5.2.1 Propagation function and Green function of a photon

The propagation function for the physical photons is obtained from the general rule (3.36):

$$K^0_{\mu\nu}(x-y) = \sum_{\lambda=1,2} \int \frac{d^3k}{(2\pi)^3} (f^{\lambda}_k)_{\mu}(x)(f^{\lambda}_k)^*_{\nu}(y) \tag{5.35}$$

Using the orthogonality condition for the photon plane waves (5.31), it is easy to check that the propagation function (5.35) indeed describes the time evolution of the freely moving photon:

$$(f^{\lambda}_k)^{\mu}(x_2) = -\int d^3 r_1 K^{\mu\nu}_0(x_2 - x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} (f^{\lambda}_k)_{\nu}(x_1) \tag{5.36}$$

Here (-) sign is due to the (-) sign in the orthogonality condition for the plane waves (5.31) [33]. Now we can construct the Green function for the photon similarly to the case of the scalar particles as the sum of the two propagation functions "forward and backward in time":

$$D^{\rm phys}_{\mu\nu}(x-y) = \Theta(x_0 - y_0) K^{\mu\nu}_0(x-y) + \Theta(y_0 - x_0) K^{\mu\nu}_0(y-x) \tag{5.37}$$

Let us check the relativistic invariance of our propagator (5.37). Similarly to the case of scalar propagator, it can be rewritten in the form of integral over 4-momenta as follows:

$$D^{\rm phys}_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4 i} \frac{1}{-k^2 - i\epsilon} \sum_{\lambda=1,2} (e^{\lambda}_k)_{\mu}(e^{\lambda}_k)_{\nu} e^{-ik(x-y)} \tag{5.38}$$

[32] The sign in the definition of the density of probability is different from the scalar case considered above since the wave functions for the photon contain space-like polarization vector e^{λ} the square of which is $e^2 = -1$

[33] This (-) sign is an artefact of the relativistic invariant notations in eq. (5.36) because in terms of spatial components of $f^{\mu} = (0, \vec{f}_i)$ the sign is $(+)$.

where we have defined transverse polarization vectors for virtual photon with $k^2 \neq 0$ as

$$\begin{aligned} e^{(1)}(k) &= (0,1,0,0) \\ e^{(2)}(k) &= (0,0,1,0) \end{aligned} \tag{5.39}$$

in the frame where $k = (k_0, 0, 0, k_3)$.

Unlike the scalar case, we do not see the relativistic invariance since the sum over the two polarizations depends on the frame for the off-mass-shell photons with $k^2 \neq 0$. Indeed, it is easy to check that

$$\sum_{\lambda=1,2} (e_k^\lambda)_\mu (e_k^\lambda)_\nu = -g_{\mu\nu} - \frac{k_\mu k_\nu}{\vec{k}^2} + \frac{k_0(k_\mu g_{\nu 0} + k_\nu g_{\mu 0})}{\vec{k}^2} - \frac{k^2 g_{\mu 0} g_{\nu 0}}{\vec{k}^2} \tag{5.40}$$

We see that only the first term is relativistic invariant while other terms are frame-dependent. In order to restore relativistic invariance we will sum over all the polarizations (not only over the physical ones). First, we define the third (longitudinal) polarization vector as

$$e^{(3)}(k) = (\frac{k_3}{\sqrt{k^2}}, 0, 0, \frac{k_0}{\sqrt{k^2}}) \tag{5.41}$$

(in the frame where $k = (k_0, 0, 0, k_3)$) so for a virtual photon with $k^2 \neq 0$ we have a set of 3 polarization vectors

$$\begin{aligned} e^{(1)}(k) &= (0,1,0,0) \\ e^{(2)}(k) &= (0,0,1,0) \\ e^{(3)}(k) &= (\frac{k_3}{\sqrt{k^2}}, 0, 0, \frac{k_0}{\sqrt{k^2}}) \end{aligned} \tag{5.42}$$

It is easy to see that $e_\mu^\lambda k^\mu = 0$ and $(e^\lambda)^2 = -1$ for any polarization vector (5.42).

If in the r.h.s. of Eq. (5.38) we sum over all the polarizations (5.42) we get

$$\sum_{\lambda=1,2} (e_k^\lambda)_\mu (e_k^\lambda)_\nu \to \sum_{\lambda=1,2,3} (e_k^\lambda)_\mu (e_k^\lambda)_\nu = -g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \tag{5.43}$$

and obtain the Green function in the so-called Landau form:

$$D^L_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4 i} \frac{1}{k^2 + i\epsilon} (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) e^{-ik(x-y)} \tag{5.44}$$

The price for the relativistic invariance of this Green function is that it describes now the propagation not only of the physical photons, but also of the unphysical longitudinal ones and we should prove in the future that in the case of the interacting theory these unphysical photons decouple from all physical amplitudes and do not contribute to S-matrix.

5.2.2 The reason for propagation of non-physical photon in electrodynamics.

Let us discuss the issue of propagation of non-physical photons. The relevant degrees of freedom for the photon are electric and magnetic fields E and H. Unfortunately, the interaction with the π-meson (or electron) field is written in terms of the vector potential A_μ rather than E and H even in the classical electrodynamics, so we are forced to deal with these vector potentials which are defined up to a gauge transformation. [34] Working in terms of A_μ we face with a problem of separation of physical degrees of freedom from the pure gauge ones (the latter are the artefacts of our description in terms of A_μ instead of $F_{\mu\nu}$). This can be done, but it turns out that if we want to have only physical photons, we should add the non-local instantaneous interaction between electrons (or π-mesons) to our interacting theory. This is clear from the form of the physical Green function which can be rewritten as follows (see eqs. (5.38) and (5.40)):

$$\begin{aligned} D_{\mu\nu}(x-y) &= D^L_{\mu\nu}(x-y) \\ &+ \int \tfrac{d^4k}{(2\pi)^4 i}\left(\tfrac{k_\mu k_\nu k_0^2}{k^4\vec{k}^2} - \tfrac{k_0(k_\mu g_{\nu 0}+k_\nu g_{\mu 0})}{k^4\vec{k}^2}\right)e^{-ik(x-y)} + \delta(x_0-y_0)\int \tfrac{d^3k}{(2\pi)^3 i}\tfrac{g_{\mu 0}g_{\nu 0}}{\vec{k}^2}e^{i\vec{k}(\vec{x}-\vec{y})} \end{aligned} \qquad (5.45)$$

The first term in the r.h.s. of this equation is the relativistic invariant Landau propagator (5.44), the second term proportional to k_μ or k_ν vanishes as we shall see in future lectures, but the last term does not vanish and leads to the instantaneous (due to $\delta(x_0-y_0)$) interaction between the particles which exchange this photon. The relativistic invariance is restored in a following way: in a consistent formulation of the quantization in this physical Coulomb gauge there exists an instantaneous Coulomb interaction which exactly cancels the contribution of the longitudinal gluons (the third term in r.h.s. of eq. (5.45)) in all Feynman diagrams. Thus, the hypothesis of locality of interactions for gauge theories must be modified in a following way: there exists a description of the theory in terms of physical and non-physical particles such that all the interactions are local. The net outcome for electrodynamics is very simple: for calculation of physical cross sections, we should use the relativistic invariant propagator (5.44) or (5.46). (In QCD the situation is more complicated: there are so-called ghost particles which appear in Feynman diagrams only in loops).

5.2.3 Feynman photon propagator

In practical calculations, it is more convenient to simplify the propagator (5.44) even more and write down the so-called Feynman propagator for the photon:

$$D^0_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4 i}\frac{g_{\mu\nu}}{k^2+i\epsilon}e^{-ik(x-y)} \qquad (5.46)$$

Again, it can be checked (and we will do that) that in the theory with electromagnetic interaction (QED) the terms proportional to the $k_\mu k_\nu$ in the r.h.s. of eq. (5.44) give vanishing contributions. (Therefore one can use the photon propagator in a more general

[34] It is possible to write down the interaction of E and H fields with charged particles, but it will be non-local, and if you start developing the theory with this non-local interaction you'll face the same problems as for the description in terms of vector potentials, only they will be hidden better and to solve them would be more difficult.

form $g_{\mu\nu} - c\frac{k_\mu k_\nu}{k^2}$ with arbitrary number c in the numerator - it will lead to the same physical results) .

Alternatively, one can proceed from the propagation function (5.35) to the Feynman propagator (5.46) in a following manner. The sum over the physical polarizations in the r.h.s of eq. (5.38) can be written as follows:

$$\sum_{\lambda=1,2} e^\lambda_\mu e^\lambda_\nu = -g_{\mu\nu} + \frac{k_\mu n_\nu + k_\nu n_\mu}{kn} \tag{5.47}$$

where $n(k)$ is the light-like vector which has the form $n = (k, 0, 0, -k)$ in the frame where $k = (k, 0, 0, k)$. (This definition fixes n up to an overall numerical factor which is unessetial since it drops from eq. (5.47)). Now, let us demand that the interaction with charged particles satisfies the condition that [35]

$$k_\mu(\text{amplitude of the production of photon with momentum k and Lorentz index } \mu) = 0 \tag{5.48}$$

Now we can drop the terms proportional to k_μ and k_ν in the propagation function. Indeed, k_μ is multiplied either by some amplitude of production of the photon with momentum k (if this photon propagation function is the internal line of the corresponding Feynman diagram) or by the polarization vector e^λ_μ (if the propagation function correspond to the external line of the Feynman diagram). In both cases, being multiplied by k_μ, they vanish [36]. Thus, if the interaction satisfies the condition (5.48) one can use the "Feynman" propagation function

$$(K_0^F)_{\mu\nu}(x-y) = -\, g_{\mu\nu} \int \frac{d^3k}{(2\pi)^3 2k_0} e^{-ikx}\Big|_{k_0=|\vec{k}|} \tag{5.49}$$

instead of the physical propagation function (5.38). If we define now the Green function in the usual form (cf. 3.47)

$$D^0_{\mu\nu} = \theta(x_0 - y_0)(K_0^F)_{\mu\nu}(x-y) + \theta(y_0 - x_0)(K_0^F)_{\mu\nu}(y-x) \tag{5.50}$$

we will get exactly the Feynman propragator (5.46).

It worth noting that the Feynman propagator (5.46) is a Green function of the d'Alembert equation in the mathematical sense: it is easy to see that

$$\Box D^0_{\mu\nu}(x-y) = i g_{\mu\nu}(2\pi)^4 \delta^{(4)}(x-y) \tag{5.51}$$

In what follows we use the photon propagator in the Feynman form.

[35] Strictly speaking, the name is not quite right since the photon has polarization λ rather than the Lorentz index μ and the amplitude of production of such photon is e^λ_μ(the rest of the amplitude)$^\mu$. What I mean here is this "rest"- the amplitude of the production of the photon stripped off its polarization vector e^λ_μ. The amplitude of emission of the photon will have a Lorentz index after such amputation.

[36] It looks like here we just imposed the condition (i): the production of any real photon vanishes after multilying by the photon momentum while above we demanded (ii): the production of any virtual photon vanishes after such multiplication. (For example, we assumed (ii) for the transition from the Landau propagator (5.44) to the Feynman one (5.46)). Careful analysis shows, however, that we need the condition (ii) for both cases (see e.g. Eq. (5.57) below). To be precise, the condition is $k_\mu \times$ [amplitude of production of virtual (or real) photon] $= 0$ if all the π-**mesons** are on the mass shell.

The reduced Green function for the photon is:

$$\mathcal{D}^0_{\mu\nu}(k) = \frac{g_{\mu\nu}}{k^2 + i\epsilon} \tag{5.52}$$

Now we have both Green functions: for the (charged) π-mesons and for the photon. In the next lecture we will construct the elementary $\pi\pi\gamma$ vertex and then the list of ingredients for the Feynman rules for the electrodynamics of π-mesons would be complete.

Part XVI

5.3 π-meson - photon interaction

Let us try to guess the form of π-meson-photon vertex from the assumption of the locality of the interaction similarly to the scalar πM interaction whih we have found in Sect. 4A. The amplitude of $\pi^+ \to \pi^+\gamma$ transition shown in Fig. 58 should have the form:

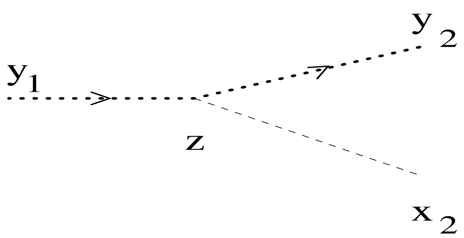


Figure 58. The $\pi^+ \Rightarrow \pi^+\gamma$ transition. The π^+ meson is denoted by the dotted line with arrow

$$G_\mu(x_2, y_2; y_1) = \int dz G_0(y_2 - z) D^0_{\mu\nu}(x_2 - z)\; \Gamma^\nu(z) G_0(z - y_1) \tag{5.53}$$

Note that the vertex should have the Lorentz index μ reflecting the dependence of the $\pi \Rightarrow \pi\gamma$ amlitude on the polarization of the emitted photon. Because of the homogenuity of the space the vertex Γ_μ should not depend on the position z. It is easy to see that the only vector which does not depend on the position z is $\frac{\partial}{\partial z^\mu}$, so the most general form of this Green function is:

$$G^0(y_2-z)\Gamma_\mu G^0(z-y_1) = aG^0(y_2-z)\frac{\partial}{\partial z^\mu}G^0(z-y_1)+b\left[\frac{\partial}{\partial z^\mu}G^0(y_2 - z)\right]G^0(z-y_1) \tag{5.54}$$

where a and b are arbitrary numbers so far. In the momentum space it has the form:

$$\mathcal{G}_\mu(p_1-k_2, k_2; p_1) = -\frac{1}{m^2 - (p_1 - k_2)^2 - i\epsilon}\frac{1}{k_2^2 + i\epsilon}(ap_{1\mu}+b(p_{1\mu}-k_{2\mu})\frac{1}{m^2 - p_1^2 - i\epsilon} \tag{5.55}$$

Now we shall demonstrare that our condition (5.48) requires that $a = b$. Let us consider the amputated Green function (5.55) with π-mesons on the mass shell:

$$\Gamma_\mu(p_1 - k_2, k_2; p_1) \stackrel{\text{def}}{\equiv} \mathcal{G}^{amp}_\mu(p_1 - k_2, k_2; p_1) = ap_{1\mu} + b(p_{1\mu} - k_{2\mu})|_{p_1^2=(p_1-k_2)^2=m^2} \tag{5.56}$$

The condition (5.48) (see also (ii) in the footnote) requires that

$$ap_1k + b(p_1 - k_2)k_2 = \frac{a-b}{2}k_2^2 = 0, \tag{5.57}$$

therefore $a = b$ in electrodynamics. Let us call this constant g for now. In principle, the numerical value of this constant should be determined from suitable scattering experiment. In practice, it is convenient to compare the result for the scattering amplitude of π-mesons (proportional to g^2) to the non-relativistic Coulomb scattering amplitude for π-mesons. We shall see that after such comparison the constant g can be identified as the electric charge of the π-meson e.

The elementary $\pi\pi\gamma$ vertex has the form

$$\Gamma_\mu(p_1 - k_2, k_2; p_1) = g(2p_1 - k_2)_\mu \tag{5.58}$$

or, in the coordinate space,

$$G_\mu(x_2, y_2; y_1) = \tag{5.59}$$
$$= \int dz G_0(y_2 - z)D^0_{\mu\nu}(x_2 - z)\Gamma^\nu(z)G_0(z - y_1) = \int dz G_0(y_2 - z)D^0_{\mu\nu}(x_2 - z)\ \overset{\leftrightarrow}{\frac{\partial}{\partial z_\nu}}\ (z)G_0(z - y_1)$$

Now, let us compare the cross section for the $\pi^+\pi^+$ elastic scattering in the Born approximation with the non-relativistic result. Calculation of the cross section for the elastic $\pi^+\pi^+$ scattering by exchange of photon practically does not differ from the calculation of the amplitude of elastic MM scattering performed in Sect. 4G. Up to the formula (4.117) we can simply copy the relevant formulas making substitution $M \to m$ when nessesary. We obtain (in the c.m. frame)

$$\frac{d\sigma}{d\Omega} = \frac{|T(p_2, p_2'; p_1, p_1')|^2}{256E_1^2\pi^2} \tag{5.60}$$

The T-matrix in the Born approximation is determined by the two diagrams shown in Fig. 59

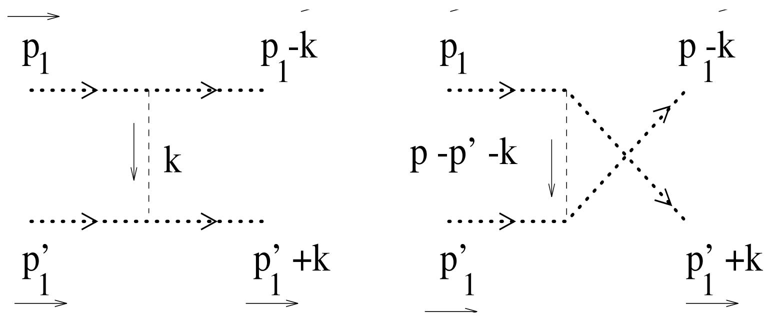


Figure 59. First-order Feynman diagrams for the elastic $\pi^+\pi^+$ scattering.

Using our expression (5.58) for the $\pi^+\pi^+\gamma$ vertex it is easy to get the following result for the reduced four-pion Green function:

$$\mathcal{G}(p_2 = p_1 - k, p_2' = p_1' + k; p_1, p_1') = \tag{5.61}$$
$$= g^2 \frac{1}{m^2 - p_1^2 - i\epsilon}\frac{1}{m^2 - p_1'^2 - i\epsilon}\frac{1}{m^2 - (p_1 - k)^2 - i\epsilon}\frac{1}{m^2 - (p_1' + k)^2 - i\epsilon}\left[\frac{(2p_1 - k)(2p_1' + k)}{k^2 + i\epsilon} + \frac{(p_1 + p_1' + k)(p_1' + p_1 - k)}{(p_1' + k - p_1)^2 + i\epsilon}\right]$$

The T-matrix is obtained by amputating the legs of this reduced Green function:

$$T(p_1-k,p_1'+k;p_1,p_1') = g^2\left(\frac{s-u}{t}+\frac{s-t}{u}\right) \tag{5.62}$$

Here we have expressed the scalar products of π-meson momenta in terms of Mandelstam variables $s=(p_1+p_1')^2,\ t=(p_2-p_1)^2,\ u=(p_2'-p_1)^2$ (see eq. (4.119)):

$$\begin{aligned}(2p_1-k)(2p_1'+k) &= s-u\\ (p_1+p_1'+k)(p_1'+p_1-k) &= s-t\end{aligned} \tag{5.63}$$

(note that now $s+t+u=4m^2$). In the c.m. frame $s=4E_1^2,\ t=-4|\vec{p}_1|^2\sin^2(\frac{\theta}{2}),\ u=-4|\vec{p}_1|^2\cos^2(\frac{\theta}{2})$ (see eq. (4.121) so our final expression for the differential elasctic $\pi^+\pi^+$ cross section (5.60) takes the form:

$$\frac{d\sigma}{d\Omega} = \frac{g^4}{256E_1^2}\left|\frac{E_1^2+|\vec{p}_1|^2\cos^2(\frac{\theta}{2})}{|\vec{p}_1|^2\sin^2(\frac{\theta}{2})}+\frac{E_1^2+|\vec{p}_1|^2\sin^2(\frac{\theta}{2})}{|\vec{p}_1|^2\cos^2(\frac{\theta}{2})}\right|^2 \tag{5.64}$$

As we mentioned above, comparing this formula to the result of the experiment on elastic $\pi\pi$ scattering can give us the value of the constant g in Nature. An easier way is to use the relevant information from the non-relativistic quantum mechanics. The non-relativistic limit of eq. (5.64) correspond to the case of small π-meson velocities ($\rightarrow E_1\simeq m$, $s\simeq 4m^2$ and $t,u\ll s$) so the expression for the cross section (5.64) reduces to

$$\frac{d\sigma}{d\Omega} = \frac{g^4m^2}{256\pi^2}\left(\frac{1}{|\vec{p}_1|^2\sin^2(\frac{\theta}{2})}+\frac{1}{|\vec{p}_1|^2\cos^2(\frac{\theta}{2})}\right)^2 \tag{5.65}$$

This should be compared with "truly" non-relativistic calculation based on the solution of Schrödinger equation for the $\pi\pi$ scattering in the Coulomb potential

$$V(\vec{r}) = \frac{e^2}{4\pi|\vec{r}|} \tag{5.66}$$

For us, the easiest way to solve this NR scattering problem is to take the cross section (4.128) of the MM scattering in Yukawa potential

$$V_Y(\vec{r}) = V_0\frac{e^{-\alpha|\vec{r}|}}{|\vec{r}|} \tag{5.67}$$

and substitute $M\rightarrow m$, $\alpha=0$, $V_0\rightarrow\frac{e^2}{4\pi}$ in eq. (4.128). We obtain

$$\frac{d\sigma}{d\Omega} = \left(\frac{me^2}{16|\vec{p}_1|^2\sin^2(\frac{\theta}{2})}+\frac{me^2}{16|\vec{p}_1|^2\cos^2(\frac{\theta}{2})}\right)^2 \tag{5.68}$$

We see now that the constant g in the $\pi\pi\gamma$ vertex can be identified with electric charge of π-meson e.

The elecric charge is small:

$$\alpha \stackrel{\text{def}}{\equiv} \frac{e^2}{4\pi} = \frac{1}{137} \tag{5.69}$$

(in the conventional units $\alpha = \frac{e^2}{2\hbar c}$) and therefore we can safely use the perturbation theory. In practice, quite often the Born approximation is enough.

Part XVII

5.4 π^+-meson and π^--meson : particle and antiparticle.

Let us consider the process of the emission of the photon by π^+-meson shown in Fig. 58. However, as we discussed many times before, one Feynman diagram corresponds to many different space-time processes. Let us consider this diagram at $x_{20} = t_2 < y_{20} = y_{10} = t_2'$ (see Fig.60).

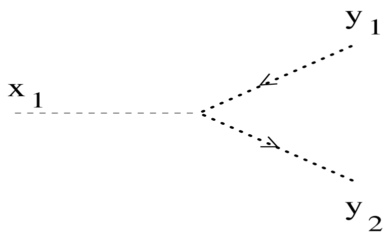


Figure 60. The "pair creation": $\gamma \Rightarrow \pi\pi$ transition.

It describes a process of transition from one-photon state at $t = t_2$ to the two-π-meson state at $t = t_2'$. However, we cannot say that the state at $t = t_2'$ is the two-π^+-meson state since it would mean non-conservation of the electric charge (we had charge 0 at the beginning of our transition and we end up with the charge 2e at $t = t_2'$). Therefore, the only way to interpret the process shown in Fig. 58 is to say that the photon creates a pair of the particle and antiparticle which have the same masses but opposite charges. Anticipating this, we have chosen a formalism where π^+-meson and π^--meson have the same mass from the beginning since they correspond to one complex scalar field. It is easy to see, however, that our arguments about antiparticles are general. Let us consider the emission of the photon by any charged particle which is described (in the Born approximation) by the diagram shown in Fig. 61a. However, one Feynman diagram describes may processes at once and in the different frame this diagram may describe a process of pair creation shown in Fig. 61b. Then, due to the charge conservation, it should be a pair of particle and antiparticle with the same masses (because in different frame it is one-particle line) and opposite charges. Thus, a charged particle must have a partner in the Nature with exactly the same mass and opposite charge follows directly from the relativistic invariance. This very important statement actually follows from the relativistic invariance and conservation of charge only (plus, of course, the hypothesis of the locality of the interaction).

It is instructive to demonstrate that the particle described by real scalar fied (π^0-meson) cannot interact with photon. Indeed , we can start building the $\pi^0\gamma$ interaction in the same way as we have done for π^+-meson in the previous Section since we have used there only the Lorentz invariance. This way we will arrive at the same vertex (5.58). Now, let

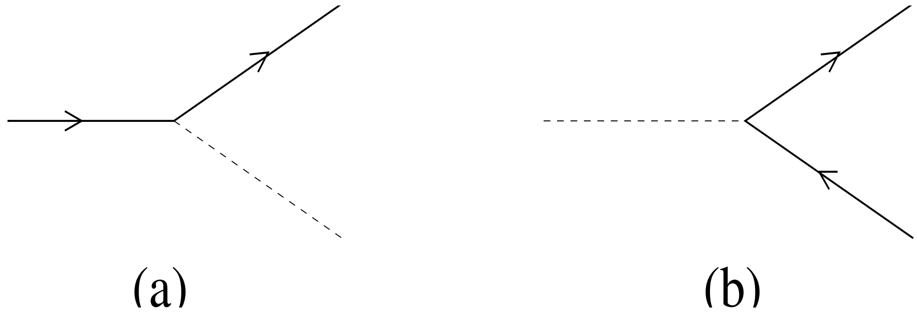


Figure 61. Two different interpretations of the same Feynman diagram: emission of the photon by a charged particle (a) and parrticle-antiparticle creation (b). A charged particle is denoted by a line with an arrow.

us show that the constant g for the $\pi^0\pi^0\gamma$ vertex should be 0. Suppose it is not. Let us consider then the state resulting from the pair creation showh in Fig. 62

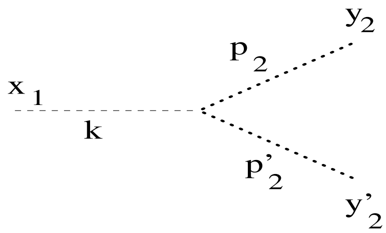


Figure 62. The gedanken $\gamma\pi^0\pi^0$ transition.

Wavefunction of this state has the form (cf. (4.89):

$$\phi_{\pi\pi}(y_2, y_2') = \int \frac{d^3p_2}{(2\pi)^3}\frac{d^3p_2'}{(2\pi)^3}\phi_{p_2}(y_2)\phi_{p_2'}(y_2')U(t_2,t_1)_{p_2,p_2';k} \tag{5.70}$$

where the evolution martix can be constructed from Green functions using our usual rules (4.70)

$$U(t_2,t_1)_{p_2,p_2';k} = \int d^3r_2 d^3r_2' d^3R_1 \phi^*_{p_2}(y_2)\phi^*_{p_2}(y_2')\, i\frac{\overleftrightarrow{d}}{dy_{20}}\, i\frac{\overleftrightarrow{d}}{dy_{20}'}\Bigg|_{y_{20}=y_{20}'=t_2} G(y_2,y_2';x_1) i\frac{\overleftrightarrow{d}}{dt_1}(f_k^\lambda)_\mu(x_1) \tag{5.71}$$

Note that this matrix element must be symmetric in momenta p_1 and p_2 as it follows from eq. (5.70) (actually, it is clear even without any formulas since the two π^0-mesons are identical).

Let us calculate this element of the U-matrix. Using our expression for the wavefunctions of the π-meson (3.26) and photon (5.30) and performing the integrations over R_1, r_2, and r'_2, we obtain:

$$\sqrt{2E_2 2E'_2 2E_k} U(t_2, t_1)_{\vec{p}_2, \vec{p}'_2; \vec{k}} =$$
$$\int \frac{dp_{20}}{2\pi} \frac{dp'_{20}}{2\pi} \frac{dk_0}{2\pi} e^{i(p_{20}+p'_{20}-E_2-E'_2)t_2}(p_{20}+E_2)(p'_{20}+E'_2)$$
$$G^\mu(p_2, p'_2; k) e^\lambda_\mu(\vec{k})(k_0+E_k) e^{i(E_k-k_0)t_1} \tag{5.72}$$

where, as usual, $E_2 = \sqrt{m^2+\vec{p}_2^2}$, $E'_2 = \sqrt{m^2+\vec{p'}_2^2}$, and $E_k = |\vec{k}|$. It is convenient to proceed from the Green function in the momentum representation to the amputated reduced Green function

$$G_\mu(p_2, p'_2; k) = -(2\pi)^3 \delta(\vec{p}_2 + \vec{p}'_2 - \vec{k})(2\pi)\delta(p_{20}+p'_{20}-k_{10})$$
$$\frac{1}{p_{20}^2 - E_2^2 + i\epsilon} \frac{1}{(p'_{20})^2-(E')_2^2+i\epsilon} \frac{1}{k_0^2-(E_k)^2+i\epsilon} \mathcal{G}^{\text{amp}}_\mu(p_1, p_2; k) \tag{5.73}$$

Then the expression for the U-matrix (5.72) reduces to:

$$\sqrt{2E_2 2E'_2 2E_k} U(t_2, t_1)_{\vec{p}_2, \vec{p}'_2; \vec{k}} = \tag{5.74}$$
$$= (2\pi)^3 \delta(\vec{p}_2 + \vec{p}'_2 - \vec{k}) \int \frac{dp_{20}}{2\pi} \frac{dk_0}{2\pi} \frac{e^{i(k_0-E_2-E'_2)t_2}}{(p_{20}-E_2-i\epsilon)(k_0-p_{20}-E'_2-i\epsilon)} \mathcal{G}^{\text{amp}}_\mu(p_2, k-p_2; k) e^\lambda_\mu(\vec{k}) \frac{e^{i(E_k-k_0)t_1}}{(k_0-E_k-i\epsilon)}$$

where the reduced amputated Green function is simply the vertex which in the Born approximation has the form (5.58) so we obtain

$$\mathcal{G}^{\text{amp}}_\mu(p_2, p'_2; k) e^\lambda_\mu(\vec{k}) = g(p_2 - p'_2)^\mu e^\lambda_\mu(\vec{k}) = -(\vec{p}_2 - \vec{p}'_2)\vec{e}^\lambda(\vec{k}) \tag{5.75}$$

Now we can perform the remaining integrations over p_{20} and k_0 which gives (at $t_2 > 0$, $t_1 < 0$):

$$U(t_2, t_1)_{\vec{p}_2, \vec{p}'_2; \vec{k}} = g \frac{(2\pi)^3 \delta(\vec{p}_2 + \vec{p}'_2 - \vec{k})}{\sqrt{2E_2 2E'_2 2E_k}} \frac{(\vec{p}_2 - \vec{p}'_2)\vec{e}^\lambda(\vec{k})}{E_k - E_2 - E'_2 - i\epsilon} \left[e^{i(E_k-E_2-E'_2)t_2} - e^{-i(E_k-E_2-E'_2)t_1}\right] \tag{5.76}$$

What is essential for us is not the explicit form of this U-matrix element but the fact that, contrary to our expectations, it is antisymmetric in $\vec{p}_2$ and $\vec{p}'_2$ as it is easy to see from eq. (5.76):

$$U(t_2, t_1)_{\vec{p}_2, \vec{p}'_2; \vec{k}} = -U(t_2, t_1)_{\vec{p}'_2, \vec{p}_2; \vec{k}} \tag{5.77}$$

This antisymmetry follows directly from the form of the vertex $\sim g(p_2 - p'_2)_\mu$ which is dictated by Lorentz invariance and gauge invariance. So, we see that the coupling of the neutral π^0-meson to a photon leads to the wrong properties of the wavefunction and therefore it should be put equal to 0. [37] On the contrary, if we have the two different particles – π^+-meson and π^--meson – nothing forbids the antisymmetric wave function.

Taking now the Feynman diagram 58 in the region where $y_{20} < y_{10}, x_{20}$ we conclude that it corresponds to the process of emission of photon by the π^--meson . If we choose the momenta as shown in Fig. 63 the vertex will be

[37] Formally, it means that if we will introduce such coupling its contribution to every process would vanish after taking into account identity of π^0-mesons so we may as well not consider it from the beginning.

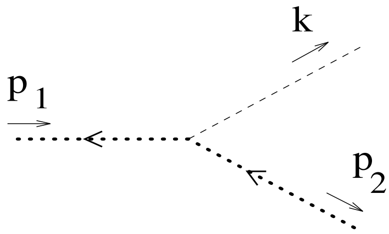


Figure 63. Emission of photon by π^--meson .

$$\Gamma_\mu(p_2 = p_1 - k, k; p_1) = e[(-p_2)_\mu + (-p_1)_\mu] = -e(p_{2\mu} + p_{1\mu}) \tag{5.78}$$

It looks like the same vertex for π^+-meson but with opposite charge. The way to remenmber the sign is that the vertex is $+$ (sum of momenta) if the direction of the arrow coincides with the chosen direction of the momentum flow.

Let us consider now the process of Compton scattering from the π^+-meson . Using our $\pi\pi\gamma$ vertex we can write down two diagrams

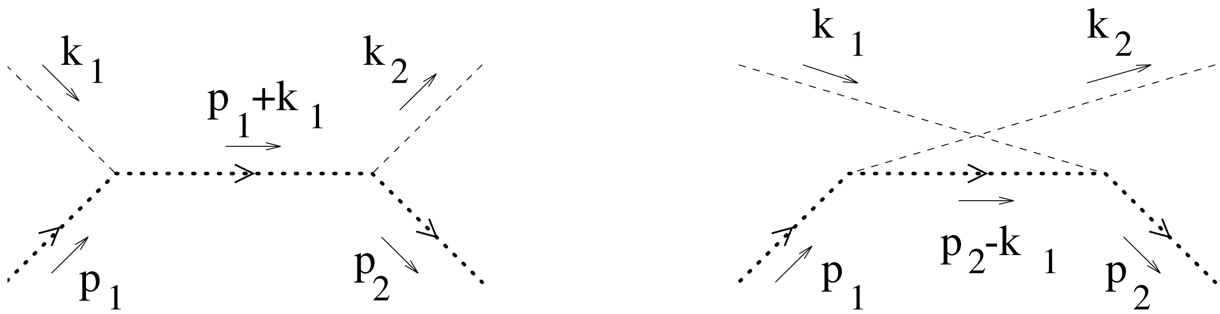


Figure 64. Two diagrams for the Compton scattering from the π^+-meson .

It is easy wo find the reduced Green function according to our rules. The amputated reduced Green function has the form:

$$\mathcal{G}^{\text{amp}}_{\mu\nu}(p_2, k_2 = p_1 + k_1 - p_2; p_1, k_1) = e^2\left(\frac{(2p_1 + k_1)_\mu(2p_2 + k_2)_\nu}{m^2 - (p_1 + k_1)^2 - i\epsilon} + \frac{(2p_1 - k_2)_\nu(2p_2 - k_1)_\mu}{m^2 - (p_1 - k_2)^2 - i\epsilon}\right) \tag{5.79}$$

Let us check now the condition (5.48) on the mass shell of the π-mesons:

$$k_1^\mu \mathcal{G}^{\text{amp}}_{\mu\nu}(p_2, k_2; p_1, k_1)\Big|_{p_1^2 = p_2^2 = m^2} = e^2 \frac{k_1(2p_1 + k_1)}{m^2 - (p_1 + k_1)^2}(2p_2 + k_2)_\nu + e^2 \frac{k_1(2p_2 - k_1)}{m^2 - (p_2 - k_1)^2}(2p_1 - k_2)_\nu \tag{5.80}$$

It is easy to see that at $p_1^2 = p_2^2 = m^2$ the numerators and denominators in r.h.s. of eq. (5.80) cancel each other so we obtain

$$k_1^\mu \mathcal{G}^{\text{amp}}_{\mu\nu}(p_2, k_2; p_1, k_1)\Big|_{p_1^2 = p_2^2 = m^2} = 2e^2(p_1 - p_2 - k_2)_\nu = -2e^2 k_{1\nu} \tag{5.81}$$

Still, this is not 0 while our condition for the interaction was that the l.h.s. of eq. (5.81) must vanish. It means that we do not have the complete understanding of our interaction -

i.e., we miss smth that should cancel this bad term. Actually, it is not difficult to find such interaction - it is “next in complexity” to our elementary emission process and it describes the contact $\pi\pi\gamma\gamma$ interaction:

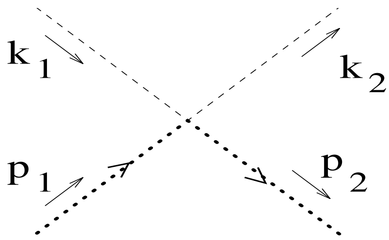


Figure 65. Contact-type for the Compton scattering from the π^+-meson .

and the corresponding vertex is simply

$$\Gamma^c_{\mu\nu}(p_2, k_2; p_1, k_1) \equiv (\mathcal{G}^{\mathrm{amp}})^c_{\mu\nu}(p_2, k_2; p_1, k_1) = 2g_{\mu\nu}e^2 \tag{5.82}$$

It is easy to see now that

$$k_1^\mu(\mathcal{G}^{\mathrm{amp}})^c_{\mu\nu} = 2k_{1\mu}e^2 \tag{5.83}$$

We see now, that eq. (5.83) exactly cancels the r.h.s. of eq. (5.81). Thus, the total set of the diagrams for the Compton scattering from the π-meson in the e^2 order is shown in Fig. 66

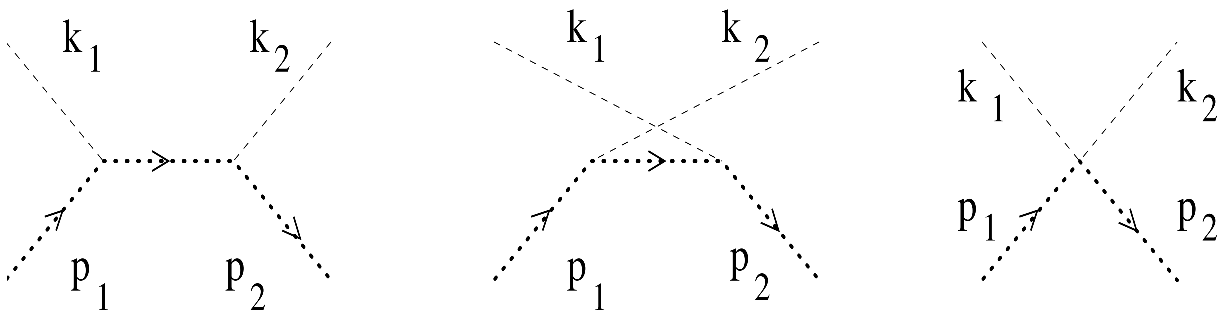


Figure 66. Total set of diagrams for $\gamma\pi \Rightarrow \gamma\pi$ scattering.

so the total contribution to the amputated reduced Green function corresponding to the sum of these three diagrams shown in Fig. 66 has the form:

$$\mathcal{G}^{\mathrm{amp}}_{\mu\nu}(p_2, k_2; p_1, k_1) = e^2\left(\frac{(2p_1+k_1)_\mu(2p_2+k_2)_\nu}{m^2-(p_1+k_1)^2-i\epsilon} + \frac{(2p_1-k_2)_\nu(2p_2-k_1)_\mu}{m^2-(p_1-k_2)^2-i\epsilon} + 2g_{\mu\nu}\right) \tag{5.84}$$

and it satisfies now the property (5.48).

5.5 Set of Feynman rules for QED of π-mesons

Feynman rules for reduced Green functions in the momentum space.

Let us summarize the Feynman rules in momentum space for the electrodynamics of π-mesons. The π-meson Green function has the form [38]:

$$\mathcal{G}_0(p) = \frac{1}{m^2 - p^2 - i\epsilon} \tag{5.85}$$

and the photon Green function in the Feynman form is

$$\mathcal{D}_{\mu\nu}(k) = \frac{g_{\mu\nu}}{k^2 + i\epsilon} \tag{5.86}$$

The vertices are

$$\begin{aligned} \pi\pi\gamma &: \Gamma_\mu(p-k, k; p) = e(2p-k)_\mu \\ \pi\pi\gamma\gamma &: \Gamma_{\mu\nu}(p', p+k-p'; p, k) = 2e^2 g_{\mu\nu} \end{aligned} \tag{5.87}$$

where it is assumed that for the $\pi\pi\gamma$ vertex the chosen direction of the flow of momentum p coincides with the direction indicated by the arrow on the π-meson line (if it is opposite, the vertex would have the opposite sign, see eq. (5.78)). [39]

The Feynman rules for the reduced Green functions are our usual rules **I-V** where we put $\mathcal{D}_{\mu\nu}(k)$ for the photon propagator and Γ_μ and $\Gamma_{\mu\nu}$ for the $\pi\pi\gamma$ and $\pi\pi\gamma\gamma$ vertices, respectively.

Feynman rules in coordinate space and combinatorial factors.

Sometimes there are non-trivial combinatorial factors for diagrams with identical photon lines. The example of such diagram with combinatorial factor $\frac{1}{2}$ is shown in Fig. 67 The contribution of this diagram has the form:

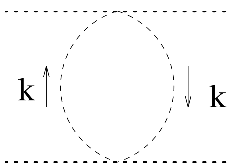


Figure 67. An example of the diagram for $\pi\pi$ scattering with the non-trivial combinatorial factor $\frac{1}{2}$

$$\mathcal{G}^{\text{amp}} = \frac{1}{2}\int \frac{d^4k}{i(2\pi)^4} 2g_{\mu\nu}\frac{g^{\nu\alpha}}{k^2+i\epsilon} 2g_{\alpha\beta}\frac{g^{be\mu}}{k^2+i\epsilon} = \frac{1}{2}\int \frac{d^4k}{i(2\pi)^4}\frac{16}{(k^2+i\epsilon)^2} \tag{5.88}$$

As usual, the easiest way to take into account these combinatorial factors is to start drawing the diagrams in the coordinate space using the initial set of Feynman rules, namely:

In order to draw all the diagrams for the Green function with m photon legs and n π-meson ones $G(x_1, x_2...x_m; y_1, y_2...y_n)$ one should perform the following steps:

[38] Note that the different sign of $(\text{momentum})^2$ in the photon propagator (5.86) is due to the fact that the sum over physical polarizations of photon is actually $-g_{\mu\nu}+$ longitudinal terms, see eq. (5.40).

[39] The $\pi\gamma$ interaction described by these vertices is called the "minimal" interaction. One can, in principle, invent more complicated interactions described by more complicated $\pi^m\gamma^n$ vertices but they are (1) non-renormalizable, and (2) do not exist in Nature.

1. Draw the $m+n$ end points (marking which of them correspond to π-mesons and which to photons).
2. Draw any number (N_1) of $\pi\pi\gamma$ vertices and any number N_2 of $\pi\pi\gamma\gamma$ vertices. Each $\pi\pi\gamma$ vertex in a point z comes with the factor $G_\mu(z) = e\,\overset{\leftrightarrow}{\frac{d}{dz_\mu}}$ where we first differentiate the π-meson Green function in the direction of the arrow on the pion line. Each $\pi\pi\gamma\gamma$ vertex comes with the factor $ie^2 g_{\mu\nu} = i\frac{1}{2}2g_{\mu\nu}$. This $\frac{1}{2}$ factor reflects identity of the two photons (compare the $\frac{1}{2}$ factor in the set of Feynman rules in coordinate space for the πM model with identical M-mesons) and there is an integration over all the space over the position of each vertex.
3. Draw all possible connections between $m+n$ end points and $N = N_1 + N_2$ vertices. Each line will be the Green function G_0 (3.50) or D_0 (5.46) depending on the type of the line. The charge of π-meson is indicated by the arrow on the line.
4. Divide the result by $N_1!N_2!$.

This $\frac{1}{N_1!N_2!}$ (and the factor 1/2 in front of each $\pi\pi\gamma\gamma$ vertex) are the combinatorial factors that will go away in the final answer - however in some cases not entirely, so it is better to keep trace of them. After performing the Fourier transformation, we return to the Feynmman rules in the momentum space described above.

Part XVIII

5.6 Cross sections in scalar QED

Let us derive the rules for getting the S-matrix elements from Green functions for the processes involving photons. These rules are actually very similar to the case of our πM model (see Sect. 4.F). For definiteness we will consider the Compton scattering discussed in previous lecture. The matrix element of the S-matrix can be written down similarly to eq. (4.93). The only difference is that we must replace, when nessesary, the wavefunctions of the scalar particles (3.26) by the photon wavefunctions (5.30) [40].

$$S^{\lambda_1,\lambda_2}(p_1,k_1 \to p_2,k_2) = \lim_{t_1\to-\infty,\ t_2\to\infty} \int d^3R_2 d^3r_2 d^3R_1 d^3r_1 \tag{5.89}$$
$$(f^{\lambda_2}_{k_2})^*_\nu(x_2)\phi^*_{p_2}(y_2)\, i\overset{\leftrightarrow}{\frac{d}{dx_{20}}}\, i\overset{\leftrightarrow}{\frac{d}{dy_{20}}}\Big|_{x_{20}=y_{20}=t_2} G^{\mu\nu}(x_2,y_2;x_1,y_1)\, i\overset{\leftrightarrow}{\frac{d}{dx_{10}}}\, i\overset{\leftrightarrow}{\frac{d}{dy_{10}}}\Big|_{x_{10}=y_{10}=t_1} (f^{\lambda_1}_{k_1})_\mu(x_1)\phi_{p_1}(y_1)$$

Similarly to the πM case (4.94) we can represent the Green function as

$$G^{\mu\nu}(x_2,y_2;x_1,y_1) = \tag{5.90}$$
$$= \int dz_2 dz_2' dz_1 dz_1' D_0^{\mu\alpha}(x_2-z_2)G_0(y_2-w_2)G^{amp}_{\alpha\beta}(z_2,w_2;z_1,w_1)D_0^{\beta\nu}(z_1-x_1)G_0(z_1'-x_1')$$

where the $G^{\mathrm{amp}}(z_2,w_2;z_1,w_1)$ is the Green function with amputated legs, see Fig. 54

Again, since $t_1 \to -\infty$, $t_2 \to \infty$ we can replace each of the Green functions G_0 and $D_0^{\xi\eta}$ in eq. (5.90) by the corresponding propagation function K_0 (see eq. (3.38)) and $K_0^{F\xi\eta}$

[40] Since the orthogonality condition for the photon wavefuctions (5.31) is essentially the same as the eq. (4.68) for the scalar plane waves, the form of the equation expressing the S-matrix element through the Green functions remains the same.

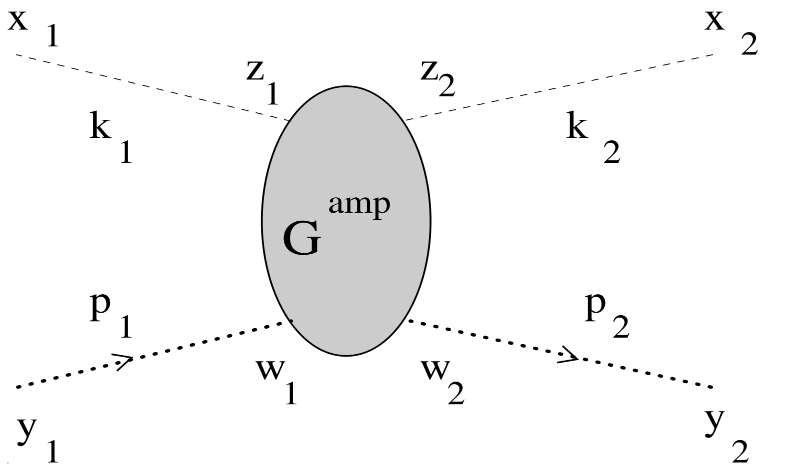


Figure 68. S-matrix from the Green function for the $\pi\gamma \Rightarrow \pi\gamma$ scattering

(see eq. (5.49)) Using the formula

$$\int d^3R_2 (f^{\lambda_2}_{k_2})^{*\mu}(x_2) i \frac{\overleftrightarrow{d}}{dx_{20}} (K^F_0)_{\mu\alpha}(x_2 - z_2) = (f^{\lambda_2}_{k_2})^*_\alpha(z_2)$$

$$\int d^3R_1 (K^F_0)_{\beta\nu}(z_1 - x_1) i \frac{\overleftrightarrow{d}}{dx_{10}} (f^{\lambda_1}_{k_1})^\nu(x_1) = (f^{\lambda_1}_{k_1})_\beta(z_1) \tag{5.91}$$

and similar formulas (4.97) for π-meson legs we can reduce the expression (4.93) to the Fourier transform of the amputated Green function times some simple factors:

$$S(p_2, p'_2; p_1, p'_1) =$$
$$= \frac{1}{\sqrt{2E_2}} \frac{(e^{\lambda_2}(k_2))^\alpha}{\sqrt{2|\vec{k}_2|}} \frac{1}{\sqrt{2E_1}} \frac{(e^{\lambda_1}(k_1))^\beta}{\sqrt{2|\vec{k}_1|}} G^{\text{amp}}_{\alpha\beta}(p_2, k_2; p_1, k_1)\Big|_{p_2^2 = p_1^2 = m^2,\ k_1^2 = k_2^2 = 0} \tag{5.92}$$

Quite similarly it can be demonstrated that the general rule for obtaining the matrix elements of the S-matrix for the transition from m_1 π-mesons with initial momenta $p_1, ..p_1^{(m_1)}$ and n_1 photons with momenta $k_1, ...k_1^{(n_1)}$ and polarizations $\lambda_1, ...\lambda_1^{(n_1)}$ to the final state of m_2 π-mesons with momenta $p_1, ..p_1^{(m_1)}$ and n_2 photons with momenta $k_1, ...k_2^{(n_2)}$ and polarizations $\lambda_2, ...\lambda_2^{(n_2)}$ has the form(cf. eq. (4.133)):

$$S^{\lambda_2,...\lambda_2^{(n_2)};\lambda_1,...\lambda_1^{(n_1)}}(p_2, ...p_2^{(m_2)}, k_2, ...k_2^{(n_2)}; p_1, ...p_1^{(m_1)}, k_1, ...k_1^{(n_1)}) = \tag{5.93}$$
$$= \prod_{j_2} \frac{1}{\sqrt{2E_2^{j_2}}} \prod_{l_2} \frac{e^{\lambda_2^{(l_2)}}_{\mu^{l_2}}(\vec{k}_2^{(l_2)})}{\sqrt{2|\vec{k}_2^{(l_2)}|}} \prod_{j_1} \frac{1}{\sqrt{2E_1^{j_1}}} \prod_{l_1} \frac{e^{\lambda_1^{(l_1)}}_{\mu^{l_1}}(\vec{k}_1^{(l_1)})}{\sqrt{2|\vec{k}_1^{(l_1)}|}}$$
$$\times\ (G^{\text{amp}})^{\mu_2,...\mu_2^{(n_2)};\mu_1,...\mu_1^{(n_1)}}(p_2, ...p_2^{(m_2)}, k_2, ...k_2^{(n_2)}; p_1, ...p_1^{(m_1)}, k_1, ...k_1^{(n_1)})\Big|_{E^{(j)} = \sqrt{(\vec{p}^{(j)})^2 + m^2}, E^{(l)} = |\vec{k}^{(l)}|}$$

So, the matrix element of S-matrix for the arbitrary $m_1 + n_1 \Rightarrow m_2 + n_2$ transition is given by the amputated Green functions on the mass shell times factors $\frac{1}{\sqrt{2E_i}}$ for each π-meson and $\frac{e^{\lambda_i}_{\mu_i}(\vec{k}_i)}{\sqrt{2E_i}}$ for each photon. Recalling the definition of the transition matrix (4.134) and the

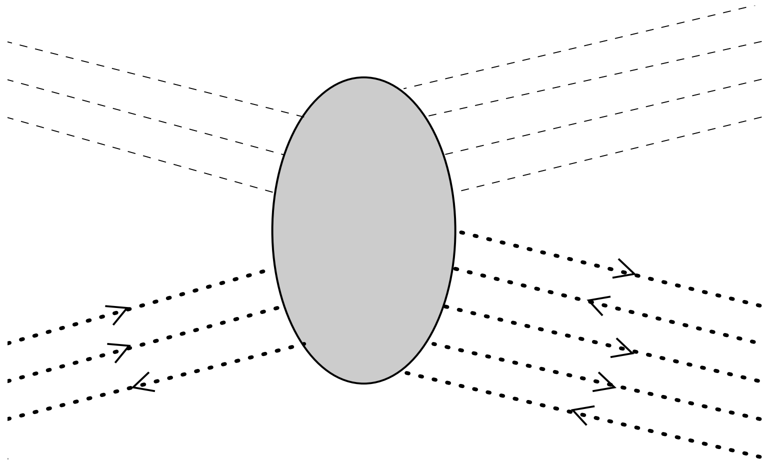


Figure 69. A general $m_1 + n_1 \Rightarrow m_2 + n_2$ scattering in QED of π-mesons

connection (4.36) between the Green function G and the reduced Green function $\mathcal{G}$ we see that the transition matrix is the amputated reduced Green function $\mathcal{G}(p_2, ...k_2^{n_2}; p_1, ..k_1^{n_1})$ on the mass shell - just as in the scalar case:

$$T^{\lambda_2,...\lambda_2^{(n_2)};\lambda_1,...\lambda_1^{(n_1)}}(p_2, ...p_2^{(m_2)}, k_2, ...k_2^{(n_2)}; p_1, ...p_1^{(m_1)}, k_1, ...k_1^{(n_1)}) =$$
$$= \prod_{l_2} e_{\mu_{(l_2)}}^{\lambda_2^{(n_2)}}(\vec{k}_2^{(l_2)}) \prod_{l_1} e_{\mu^{l_1}}^{\lambda_1^{(l_1)}}(\vec{k}_1^{(n_1)})$$
$$(\mathcal{G}^{\text{amp}})^{\mu_2,...\mu_2^{(n_2)};\mu_1,...\mu_1^{(n_1)}}(p_2, ...p_2^{(m_2)}, k_2, ...k_2^{(n_2)}; p_1, ...p_1^{(m_1)}, k_1, ...k_1^{(n_1)})\Big|_{p_i^2=m^2, k_i^2=0} \tag{5.94}$$

This is the general expression for the transition matrix in the scalar eletrodynamics. Together with Feynman rules for the reduced Green functions it gives a complete set of prescriptions for calculating of the probability of any transition in the electrodynamics of π-mesons. The cross section for the arbitrary $2 \Rightarrow m + n$ scattering in terms of T-matrix looks exactly the same as for scalar particles (4.141) except for trivial differences in masses (and hence different flux (4.109). The same is true also for the optical theorem (4.147).

Homework assignment 4.
Find the differential cross section for the $\pi^+\pi^- \Rightarrow \pi^+\pi^-$ scattering in the first nontrivial order in perturbation theory.

Part XIX

5.7 Compton scattering , $\pi^+\pi^-$-annihilation and crossing symmetry

Let us finish now our calculation of the cross section for the Compton scattering from the π^+-meson in the first order in $\alpha \equiv \frac{e^2}{4\pi}$. The corresponding diagrams were shown in Fig. 66. The amputated reduced Green function is given by eq. (5.84) so the relevant element of the T-matrix has the form:

$$T^{\lambda_1\lambda_2}(p_2,k_2;p_1,k_1) =$$
$$e^2\left(\frac{4(e^{\lambda_1}(k_1)\cdot p_1)(e^{\lambda_2}(k_2)\cdot p_2)}{m^2-(p_1+k_1)^2-i\epsilon}+\frac{4(e^{\lambda_2}(k_2)\cdot p_1)(e^{\lambda_1}(k_1)\cdot p_2)}{m^2-(p_1-k_2)^2-i\epsilon}+2e^{\lambda_2}(k_2)\cdot e^{\lambda_1}(k_1)\right) \tag{5.95}$$

where we used the fact that $e(k_i)\cdot k_i = 0$. From the HW3 and Appendix A we know that the cross section of the elasctic two-particle scattering in the c.m. frame has the form:

$$\frac{d\sigma}{d\Omega} = \frac{|T|^2}{64\pi^2 s} \tag{5.96}$$

To calculate the T-matrix (5.95) let us choose the polarization vectors as shown in Fig. 70 [41].

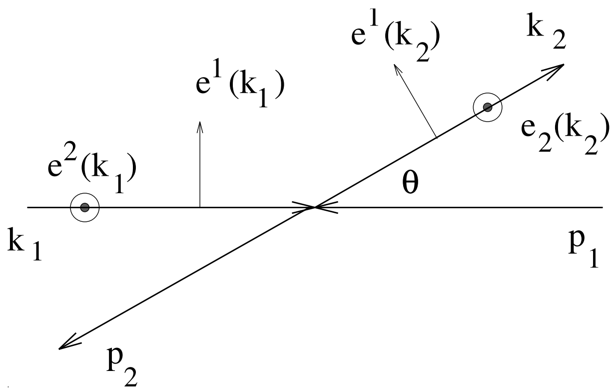


Figure 70. Kinematics for Compton scattering in the c.m. frame

It is instructive to present the result in terms of the Mandelstam variables (4.119). Using the formulas:

$$\begin{aligned}
|\vec{p_i}| = \quad & |\vec{k_i}| &= \frac{s-m^2}{2\sqrt{s}}\\
e^{(i)}(k_1)\cdot p_1 = e^{(i)}(k_2)\cdot p_2 &= e^{(2)}(k_2)\cdot p_1 = e^{(2)}(k_1)\cdot p_2 = 0\\
-e^{(1)}(k_1)\cdot p_2 = e^{(1)}(k_2)\cdot p_1 &= \frac{s-m^2}{2\sqrt{s}}\sin\theta\\
m^2-(p_2-k_1)^2 = \quad 2p_2k_1 &= \frac{(s-m^2)^2}{2s}\left(\frac{s+m^2}{s-m^2}+\cos\theta\right)
\end{aligned} \tag{5.97}$$

we easily obtain

$$\begin{aligned}
T^{1,2}(s,\theta) &= T^{2,1}(s,\theta) = 0, \qquad T^{2,2} = -2e^2,\\
T^{1,1} &= -2e^2\Big[-\frac{2|\vec{p}|^2\sin^2\theta}{u-m^2}+\cos\theta\Big] = -2e^2\Big[1+\frac{2st}{(s-m^2)^2}+\frac{2t}{u-m^2}+\frac{2st^2}{(s-m^2)(u-m^2)}\Big]\\
&= -2e^2\Big[1+\frac{2t}{u-m^2}+\frac{2st(u+t-m^2)}{(s-m^2)(u-m^2)}\Big] = -2e^2\Big[1-\frac{2tm^2}{(u-m^2)(s-m^2)}\Big]
\end{aligned} \tag{5.98}$$

[41] One may note that it is inconvenient to choose the initial polarization vectors e^1 and e^2 depending on the direction of the final momenta (as it was actually done in Fig. 70 where the final momenta determines the plane orthogonal to $e^{(2)}(k_1)$). However, we will use this choice of $e^{(i)}$ as an intermediate step to obtains the *helicity amplitudes* (defined below) which are independent of the choice of polarization vectors.

since $s+t+u=2m^2$ and $\cos\theta = 1+\frac{2ts}{(s-m^2)^2}$. Thus, the transition matrix takes the form

$$T^{\lambda_1,\lambda_2}(s,t) = -2e^2\delta_{\lambda_1\lambda_2}\left[\delta_{1\lambda_1}\left(1-\frac{2tm^2}{(u-m^2)(s-m^2)}\right)+\delta_{2\lambda_1}\right] \tag{5.99}$$

In terms of energy and angle θ it reads as

$$T^{\lambda_1,\lambda_2}(s,\theta) = -2e^2\delta_{\lambda_1\lambda_2}\left[\delta_{1\lambda_1}\left(\cos\theta+\frac{\sin^2\theta}{\frac{s+m^2}{s-m^2}+\cos\theta}\right)+\delta_{2\lambda_1}\right] \tag{5.100}$$

In the high-energy physics often the results are expressed in terms of so-called helicity amplitudes. Let us define two circular polarization vectors:

$$\vec{e}^{\,+}(k) = \frac{\vec{e}^{\,(1)}(k)+i\vec{e}^{\,(2)}(k)}{\sqrt{2}}; \quad \vec{e}^{\,-}(k) = \frac{\vec{e}^{\,(1)}(k)-i\vec{e}^{\,(2)}(k)}{\sqrt{2}} \tag{5.101}$$

These vectors are light-like: $(e^+)^2 = (e^-)^2 = 0$ and the normalization condition is $e^-e^+ = 1$ (note that $(e^+)^* = e^-$). The helicity is defined as the component of the spin along the direction of the momentum of the particle, see Fig. 71.

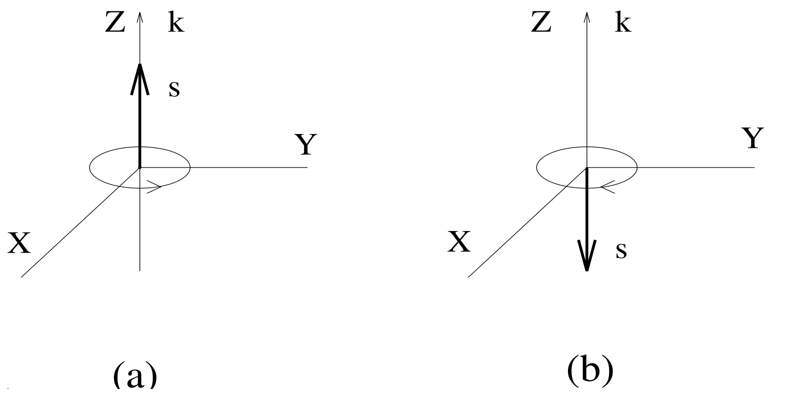


Figure 71. Photon with positive (a) and negative (b) helicity.

Suppose we make a rotation on the angle ϕ around the axis OZ which we choose to be parallel to the vector $\vec{k}$. The components of any 4-vector a are transformed as follows:

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{5.102}$$

where the non-trivial block is simply a matrix of rotation on the angle ϕ in XY-plane. Now, let us check how the circular polarization vectors (5.101) transform under this rotation. Let us start from the x-component of the vector e^+. Under the rotation (5.102) it transforms as follows:

$$e_x^+ = e_x^{(1)} + ie_x^{(2)} \to [e_x^{(1)}\cos\phi - e_y^{(1)}\sin\phi] + i[e_x^{(2)}\cos\phi - e_y^{(2)}\sin\phi] \tag{5.103}$$

Since vectors $e^{(1)}$ and $e^{(2)}$ are orthogonal $e_x^{(1)} = e_y^{(2)}$ and $e_x^{(2)} = -e_y^{(1)}$ so we get

$$\begin{aligned} e_x^+ &= [e_x^{(1)} \cos\phi + e_x^{(2)} \sin\phi] + i[e_x^{(2)} \cos\phi - e_x^{(1)} \sin\phi] \\ &\to e_x^{(1)}(\cos\phi - i\sin\phi) + ie_x^{(2)}(\cos\phi - i\sin\phi) \to e_x^+ e^{-i\phi} \end{aligned} \tag{5.104}$$

Similarly, one can show that $e_y^+ \to e_y^+ e^{-i\phi}$, $e_x^- \to e_x^- e^{i\phi}$, and $e_y^- \to e_y^- e^{i\phi}$. Thus, the circular polarization vectors transform under the rotations around the $OZ \parallel \vec{k}$ direction as follows:

$$e^+ \to e^{-i\phi} e^+, \quad e^- \to e^{i\phi} e^- \tag{5.105}$$

and since this rotation does not affect the photon momentum $\vec{k} \parallel OZ$, the same will be true for the photon wavefunctions $(f_k^{(+)})_\mu = \frac{e_\mu^+(k)}{\sqrt{2E_k}} e^{-ikx}$ and $(f_k^{(-)})_\mu = \frac{e_\mu^-(k)}{\sqrt{2E_k}} e^{-ikx}$:

$$\begin{aligned} (f_k^{(+)})_\mu &\to e^{-i\phi}(f_k^{(+)})_\mu, \\ (f_k^{(-)})_\mu &\to e^{i\phi}(f_k^{(-)})_\mu \end{aligned} \tag{5.106}$$

From the quantum mechanics we (should) remember that the state has a the projection s_z of the spin on the OZ axis if under the active rotation (see the footnote on page 113) on the angle ϕ around this axis the wavefunction is multiplied by phase factor proportional to ϕ with the coefficient of the proportionality s_z being the z-component of the spin:

$$\langle \phi | \Psi \rangle \to e^{-i\phi s_z} \langle \phi | \Psi \rangle \tag{5.107}$$

So we see that the helicity ($\equiv$ projection of the photon spin on the direction of the photon momentum) is $+1$ for the circular polarization (e^+) and -1 for the circular polarization (e^-) [42].

Then the T-matrix for the transition between the states with definite helicities T^{h_1,h_2} has the form:

$$\begin{aligned} T^{++} &= (e_\mu^+(k_2))^* e_\nu^+(k_1)(\mathcal{G}^{\text{amp}})^{\mu\nu}(p_2,k_2;p_1,k_1) = \frac{1}{2}(T^{11} + T^{22}) \\ T^{--} &= (e_\mu^-(k_2))^* e_\nu^-(k_1)(\mathcal{G}^{\text{amp}})^{\mu\nu}(p_2,k_2;p_1,k_1) = \frac{1}{2}(T^{11} + T^{22}) \\ T^{-+} &= (e_\mu^-(k_2))^* e_\nu^+(k_1)(\mathcal{G}^{\text{amp}})^{\mu\nu}(p_2,k_2;p_1,k_1) = \frac{1}{2}(T^{11} - T^{22}) \\ T^{+-} &= (e_\mu^+(k_2))^* e_\nu^-(k_1)(\mathcal{G}^{\text{amp}})^{\mu\nu}(p_2,k_2;p_1,k_1) = \frac{1}{2}(T^{11} - T^{22}) \end{aligned} \tag{5.109}$$

[42] It is worth noting that for a massive particle with spin 1 (like Z_0 boson which is a "massive photon" in a sense) there are three possible components of the projection of the spin on the direction of momentum: $+1, -1$, and 0. For the massless particles, however,we see that the third opportunity is missing. It is related to the fact that for the massive vector particles with mass μ we have the three polarization vectors (which in the frame where $k = (k_0, 0, k_3)$ can be chosen as :

$$e^{(1)}(k) = (0,1,0,0), \quad e^{(2)}(k) = (0,0,1,0), \quad e^{(3)} = (k_3/\mu, 0, 0, k_0/\mu) \tag{5.108}$$

cf. eq. (5.42)) while in the case of massless particle the third opportunity corresponding to the helicity 0 is actually the unphysical gauge degree of freedom.

Thus, we obtain

$$T^{++}(s,t) = T^{--}(s,t) = -2e^2\left(1 - \frac{tm^2}{(u-m^2)(s-m^2)}\right) = -e^2\Big(1+\cos\theta + \frac{\sin^2\theta}{\frac{s+m^2}{s-m^2}+\cos\theta}\Big)$$
$$T^{+-}(s,t) = T^{-+}(s,t) = -2e^2\left(-\frac{tm^2}{(u-m^2)(s-m^2)}\right) = e^2\Big(1-\cos\theta - \frac{\sin^2\theta}{\frac{s+m^2}{s-m^2}+\cos\theta}\Big) \tag{5.110}$$

and using the formula (5.96) we can easily get the relevant cross sections:

$$\left(\frac{d\sigma}{d\Omega}\right)^{++} = \left(\frac{d\sigma}{d\Omega}\right)^{--} = \frac{e^4}{16\pi^2 s}\left(1+\cos\theta + \frac{\sin^2\theta}{\frac{s+m^2}{s-m^2}+\cos\theta}\right)^2$$
$$\left(\frac{d\sigma}{d\Omega}\right)^{+-} = \left(\frac{d\sigma}{d\Omega}\right)^{-+} = \frac{e^4}{16\pi^2 s}\left(1-\cos\theta - \frac{\sin^2\theta}{\frac{s+m^2}{s-m^2}+\cos\theta}\right)^2 \tag{5.111}$$

The advantage of the helicity amplitudes (5.109) is that they are relativistic invariant up to an overall phase factor (which disappears if we calculate the cross sections). Let us prove it. Suppose we have the circular polarization vectors $e^{\pm}(\vec{k}) = (e^{(1)}(\vec{k}) \pm ie^{(2)}(\vec{k}))/\sqrt{2}$ in a certain frame. If we perform a Lorentz boost, the Lorentz transforms $L(e^{(1)})$ and $L(e^{(2)})$ are no longer the physical polarization vectors in a new frame where the photon momentum is $L(k) \equiv k'$ (for example, they have the time component while the vectors of physical polarization should not have it). However, we can choose new physical polarization vectors $e^{(i)}_{\rm new}(k')$ in such a way that that $L(e^{(1)}) = e^{(1)}_{\rm new}(k') + const \cdot k'_\mu$ and similarly for $e^{(2)}$ [43]. Since the multiplication of $T^{\mu\nu}$ by k'_μ vanishes due to Ward identity we get $e^{(i)}_\mu T^{\mu\nu} e^{(j)}_\nu = (e^{(i)}_{\rm new})_\mu T^{\mu\nu}(e^{(j)}_{\rm new})_\nu$. In general, the choice of physical polarization vectors for k' may differ from $e^{(i)}_{\rm new}(k')$ by the overall rotation around the direction of the momentum and therefore the corresponding circular polarization vectors differ by a phase factor

$$e^{(\pm)}(k') = e^{(\pm)}_{\rm new}(k')e^{\pm i\phi} \tag{5.113}$$

where the phase ϕ depends on our concrete choice of physical polarization vectors in two frames. Therefore, when we change the frame we get

$$T^{++} \to T^{++}, \quad T^{--} \to T^{--}, \quad T^{+-} \to e^{2i\phi}T^{+-}, \quad T^{-+} \to e^{-2i\phi}T^{-+} \tag{5.114}$$

The cross sections with definite helicities are proportional to $|T|^2$ so they are relativistic invariant.

[43] We can choose

$$\vec{e}^{(1)}_{\rm new} = L(\vec{e}^{(1)}) - \frac{L(\vec{e}^{(1)})_0}{k'_0}\vec{k}'$$
$$\vec{e}^{(2)}_{\rm new} = L(\vec{e}^{(2)}) - \frac{L(\vec{e}^{(2)})_0}{k'_0}\vec{k}' \tag{5.112}$$

It is easy to check that $(e^{(i)}_{\rm new})^2 = -1$, $(\vec{e}^{(1)}_{\rm new}\cdot\vec{e}^{(2)}_{\rm new}) = 0$, $(\vec{e}^{(i)}_{\rm new}\cdot\vec{k}') = 0$, and $(\vec{e}^{(i)}_{\rm new})_0 = 0$ (by construction) so they can serve as a physical polarization vectors for the photon with momentum k'.

If we are making an experiment with initial photon beam polarized in some fixed direction (say, along x axis while the momentum in c.m. frame is along the z axis) we must at first decompose in helicity amplitudes, and then calculate the cross section $\sim |T|^2$. For example, let the initial photon be polarized along the x axis while the direction of the momentum in c.m. frame is along the z axis and supppose we are interested in positive-helicity photon in the final state (see Fig. 72). Then first we write

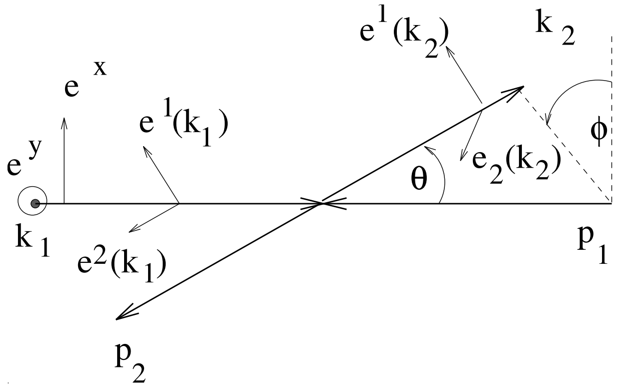


Figure 72. Asimuthal dependence for Compton scattering in the c.m. frame

$$\vec{e}^{(x)} = \frac{1}{\sqrt{2}}(\vec{\epsilon}^{+} + \vec{\epsilon}^{-}) \tag{5.115}$$

where $\vec{\epsilon}^{\pm} = \frac{1}{\sqrt{2}}(\vec{e}^{x} \pm i\vec{e}^{y})$. Second, we must relate our circular vectors ϵ^+ and ϵ^- to the similar vectors e^+ and e^- but defined according to the picture in the Fig. 70 which means that they are related by the rotation on the asimuthal angle ϕ around the $OZ \parallel \vec{k}_1$ axis. Fortunately, for the circular polarizations corresponding to definite helicity the rotation properties are trivial:

$$\epsilon^{\pm} = e^{\pm i\phi} e^{\pm} \tag{5.116}$$

(see eq. (5.105)) so the relevant amplitude of transition from the polarization e^x to positive helicity in the final state is

$$T^{x+} = e^{x}_{\mu} T^{\mu +} = \frac{1}{\sqrt{2}}(e^{+}e^{i\phi} + e^{-}e^{-i\phi})_{\mu} T^{\mu +} = \frac{1}{\sqrt{2}}(e^{i\phi}T^{++} + e^{-i\phi}T^{-+}) \tag{5.117}$$

This is enough for calculation of the cross section:

$$\frac{d\sigma^{x+}}{d\Omega} = \frac{1}{64\pi^2 s}\left(|T^{++}|^2 + |T^{-+}|^2 + 2\cos 2\phi |T^{++}T^{-+}|\right) \tag{5.118}$$

So, using the helicity amplitudes, we can easily take into account the dependence on the asimuthal angle (if we set so our polarization experiment).

Sometimes it is useful to know the cross section calculated in the rest frame of the π-meson . The initial 4-momentum of the pion is then ($p_1 = (m, 0, 0, 0)$) and suppose it is striked by a photon moving along the Z axis with momentum $k_1 = (|\vec{k}_1|, 0, 0, \vec{k}_1)$. In this case the expression for the differential cross section has the form:

$$\left(\frac{d\sigma}{d\Omega}\right)^{\text{Compton}}_{\text{lab}} = \frac{|T|^2}{64\pi^2}\frac{1}{[m+|\vec{k}_1|(1-\cos\theta)]^2} \tag{5.119}$$

(see Appendix B) and the corresponding T-matrix element is:

$$\begin{aligned} T^{++}(s,t) = T^{--}(s,t) &= -2e^2\left(1-\frac{tm^2}{(u-m^2)(s-m^2)}\right) = -e^2(1+\cos\theta) \\ T^{+-}(s,t) = T^{-+}(s,t) &= -2e^2\left(-\frac{tm^2}{(u-m^2)(s-m^2)}\right) = e^2(1-\cos\theta) \end{aligned} \tag{5.120}$$

where we have used the fact that the helicity amplitudes are relativistic invariant and plugged in the explicit expressions for Mandelstam variables in the lab frame (8.17) [44].

Homework assignment 5.
Find the T-matrix for the Compton scattering in the lab frame in the leading order in α by direct calculation of r.h.s. of eq. (5.95) in the lab frame. For simplicity, choose polarization vectors as in Fig. 90.

5.7.1 $\pi^+\pi^-$-annihilation and crossing symmetry.

Let us now conside the process of $\pi^+\pi^-$-annihilation into a pair of photons. The relevant diagrams in the first order in α are shown in Fig. 73. Let us choose the polarization vectors

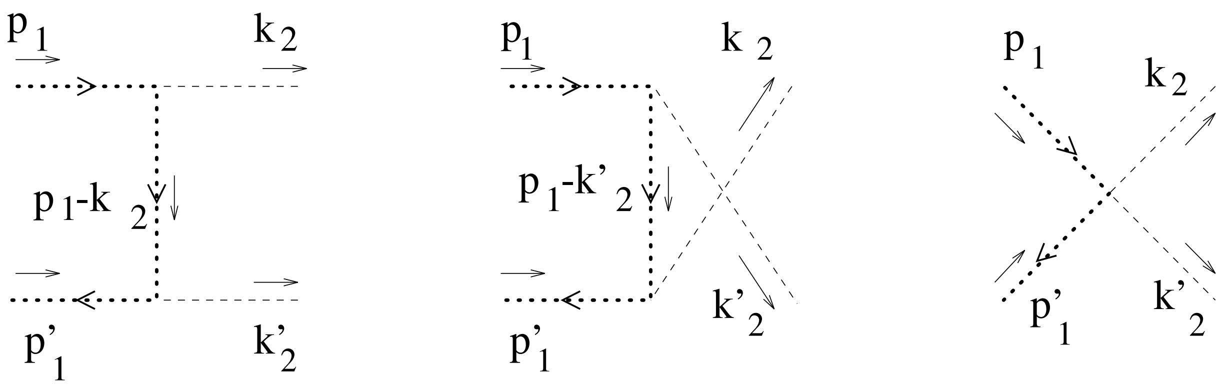


Figure 73. Feynman diagrams for $\pi^+\pi^-$-annihilation .

in the c.m. frame as shown in Fig. 73

Using our Feynman rules, it is easy to write down the reduced Green function for this process:

$$\begin{aligned} &(\mathcal{G}^{\text{amp}})^{\mu\nu}(k_2,k_2';p_1,p_2) = \\ &e^2\left(\tfrac{(2p_1-k_2)_\mu(-2p_1'+k_2')_\nu}{m^2-(p_1-k_2)^2-i\epsilon}+\tfrac{(2p_1-k_2')_\nu(-2p_1'+k_2)_\mu}{m^2-(p_1-k_2')^2-i\epsilon}+2g_{\mu\nu}\right) \end{aligned} \tag{5.121}$$

[44] In general, there may be a phase factor $e^{i\phi}$ in the expressions fot $T^{+-}$and T^{-+} (see eq. (5.114) but if we choose the polarization vectors as shown in Fig.91 this phase factor is absent.

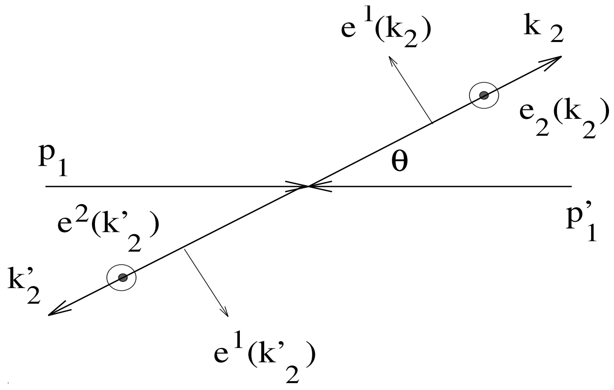


Figure 74. Kinematics for $\pi^+\pi^-$-annihilation into photons in the c.m. frame

Actually, it is the same amplitude (5.84) only with the renamed momenta:

$$(\mathcal{G}^{\rm amp})^{\mu\nu}_{\rm annihilation}(k_2,k_2';p_1,p_2)= \\ (\mathcal{G}^{\rm amp})^{\mu\nu}_{\rm Compton}(p_2\to -p_1',k_2\to k_1';p_1\to p_1,k_1\to -k_1) \tag{5.122}$$

The T-matrix for the $\pi^+\pi^-$-annihilation has the form:

$$T^{\lambda_2\lambda_2'}(k_2,k_2';p_1,p_1')= \\ e^2\left(-\frac{4(e^{\lambda_2}(k_2)\cdot p_1)(e^{\lambda_2'}(k_2')\cdot p_1')}{m^2-(p_1-k_2)^2-i\epsilon}-\frac{4(e^{\lambda_2'}(k_2')\cdot p_1)(e^{\lambda_2}(k_2)\cdot p_1')}{m^2-(p_1-k_2')^2-i\epsilon}+2e^{\lambda_2'}(k_2')\cdot e^{\lambda_2}(k_2)\right) \tag{5.123}$$

Using the formulas $e^{(2)}(k_i)\cdot p_j=e^{(2)}(k_i)\cdot k_j=0$ and $|\vec{p}_1|=|\vec{p'}_1|=\frac{1}{2}\sqrt{s-4m^2}$ we get

$$e^{(1)}(k_2)\cdot p_1=-e^{(1)}(k_2)\cdot p_1'=-e^{(1)}(k_2')\cdot p_1=e^{(1)}(k_2')\cdot p_1'=\frac{1}{2}\sqrt{s-4m^2}\sin\theta \tag{5.124}$$

and therefore

$$\begin{aligned} T^{12} &= T^{21} = 0, \quad T^{22} = -2 \\ T^{11} &= e^2\Big(2-\frac{s(s-4m^2)}{(m^2-u)(m^2-t)}\sin^2\theta\Big) = -2\Big[1-\frac{2m^2s}{(m^2-t)(m^2-u)}\Big] \end{aligned} \tag{5.125}$$

where we have used the formula $u-t=4|\vec{p}_1||\vec{k}_2|\cos\theta\Rightarrow$

$$\cos^2\theta = \frac{(u-t)^2}{s(s-4m^2)} \Rightarrow \sin^2\theta=\frac{4(m^2-t)(m^2-u)-4m^2s}{s(s-4m^2)} \tag{5.126}$$

Thus, one gets

$$T^{\lambda_2,\lambda_2'}=-2\delta_{\lambda_2\lambda_2'}\left(\delta_{1\lambda_2}\left(1-\frac{2m^2s}{(t-m^2)(u-m^2)}\right)+\delta_{2\lambda_2'}\right) \tag{5.127}$$

In terms of energy and angles it reads as

$$T^{\lambda_2,\lambda_2'}=-2e^2\delta_{\lambda_2\lambda_2'}\Big[\delta_{1\lambda_2}\Big(1-\frac{2}{\cos^2\theta+\frac{s}{4m^2}\sin^2\theta}\Big)+\delta_{2\lambda_2}\Big] \tag{5.128}$$

Finally, let us present the helicity amplitudes:

$$T^{+-}(s,t) = T^{-+}(s,t) = \frac{1}{2}(T^{11} + T^{22}) = -2e^2\left(1 - \frac{sm^2}{(u-m^2)(t-m^2)}\right)$$
$$T^{++}(s,t) = T^{--}(s,t) = \frac{1}{2}(T^{11} - T^{22}) = -2e^2\left(-\frac{sm^2}{(u-m^2)(t-m^2)}\right) \tag{5.129}$$

It is easy to see that (5.129) coincide with the T-matrix (5.110) for the Compton scattering after change $s \leftrightarrow t$ [45]:

$$T^{h_2,h_2'}_{\text{annihilation}}(s,t,u) = T^{-h_2',h_2}_{\text{Compton}}(t,s,u) \tag{5.130}$$

where h, h' are the helicities of the photons ($+$ or $-$). (The change of helicity is due to the fact that we multiply by $e^{*\lambda_2'}(k_2')$ in the final state but by $e^{\lambda_1}(k_1)$ in the initial state). This property is called crossing symmetry - the transition amplitude for the particle-antiparticle annihilation coincide with the amplitude of Compton scattering of the photon from the particle. Note, however, that this is an unphysical region for the Compton scattering since the initial photon and final π^+-meson has negative energy. So, it is better to say that the amplitudes of annihilation and Compton scattering are related by analytical continuation. In the first order in perturbation theory considered above this statement may apppear trivial, but nontrivial fact is that the crossing symmetry is exact - it is valid in all orders in perturbation theory and even in the theories with strong interactions such as QCD where it is related to the so-called CPT theorem. In general, the crossing symmmetry reads

$$T(A + B \Rightarrow C + D) = T(A + \bar{C} \Rightarrow \bar{B} + D) \tag{5.131}$$

where $\bar{B}$ means antiparticle (with 4-momentum $=\ -$ momentum of the particle).

Now that we know the T-matrix element, it is easy to write down the differential cross section in the c.m. frame. We can start with the expression (4.112).

$$d\sigma = \frac{1}{I}\frac{d^3k_2}{(2\pi)^3 4E_2E_2'} 2\pi\delta(E_2 + E_2' - E_1 - E_1')|T(k_2,k_2';p_1,p_1')|^2 \tag{5.132}$$

Now, in the c.m. frame $\vec{k}_2 = \vec{k}_2'$ ($\Rightarrow E_2 = E_2'$) and also $\vec{p}_1 = \vec{p}_1'$ ($\Rightarrow E_1 = E_1'$) but for the $\pi^+\pi^-$-annihilation $|\vec{k}_2| \neq |\vec{p}_1|$. Pefroming the integration over $|\vec{k}_2|$ with the help of δ -function one obtains:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\sqrt{\frac{s}{s-4m^2}}|T^{h_2,h_2'}(s,\theta)|^2 \tag{5.133}$$

and together with eq. (5.129) it gives us the answer for the differential cross section of $\pi^+\pi^-$-annihilation into photons.

[45] It is very convenient to compare the helicity amplitudes (5.109) since they do not depend on our particular choice of $e^{(1)}$ and $e^{(2)}$ for the initial or final photons

Part XX

5.8 Ward identity

Now that we know how to calculate S-matrix elements in scalar electrodynamics we can return to the origins and prove the property which was promised in Section 5C - proof that we do not produce the unphysical photons in the collisions of physical particles. In principle, there is a danger of such production since in order to make our Green fuction relativistic invariant we have added to the terms describing the propagation of physical transverse photons (see eq. (5.38)) the terms corresponding to the propagation of the unphysical longitudinal photons and so th nonphysical particles are present in our diagrams - but not in the S-matrix, as we will show in this Section.

First, let us prove the Ward identity (5.48) which can be carefully formulated as follows:

Ward identity:

Suppose we have a general amputated Green function [46] $G^{\rm amp}_{\mu_1,...,\mu_m}(k_1,...k_m,p_1,...p_l)$ with all the π-meson momenta $p_1,...p_l$ on the mass shell ($p_i^2=m^2$). Then

$$k_i^{\mu_i}G^{\rm amp}_{\mu_1,...,\mu_m}(k_1,...k_m,p_1,...p_l)=0 \tag{5.134}$$

We will prove the Ward identity (5.134) order by order in e, by looking directly at the Feynman diagrams that contribute to G. The identity is generally not true for individual Feynman diagrams; we must sum over the diagrams for G at any given order.

Consider a typical diagram for a typical amputated Green function $G^{(N)}$ (where N is the order in perturbation theory $\equiv$ number of vertices) shown in Fig. 75

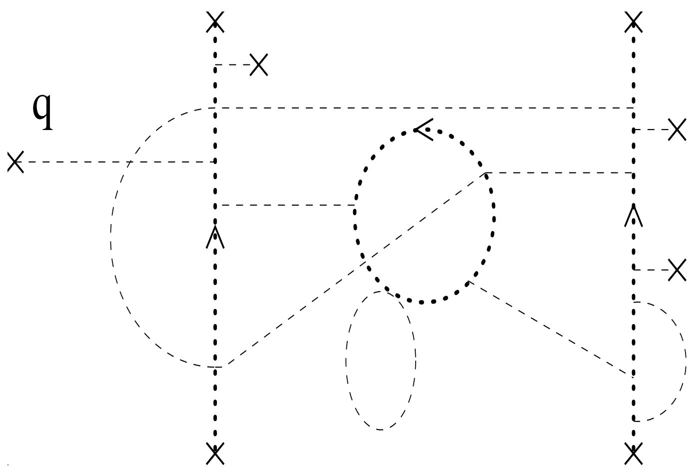


Figure 75. A typical Feynman diagram for scalar QED.

If we remove the photon with momentum k_i, which we will call q from now on, we obtain a simpler diagram for the (amputated) Green function $G^{(N-1)}$. If we reinsert this

[46] Note that usually the name "Ward identity" is reserved for a more general formula relating different off-shell Green functions of which our property (5.134) is a consequence.

photon somewhere else iside the simpler diagram, we again obtain a contribution to $G^{(N)}$. The crucial observation is that by summing over all the diagrams that contribute to $G^{(N-1)}$, then summing over all the possible points of the insertion in each of these diagrams, we get back the original Green function $G^{(N)}$. The Ward identity (5.134) is true individually for each diagram contributing to $G^{(N-1)}$; this is what we will prove.

When we insert our photon into one of the diagrams of $G^{(N-1)}$, it must attach either to an electron line that runs out of the diagram to two external points, ot to an internal π-meson loop. Let us these opprtunities in turn. In each case, we will consider a typical example the general proof can be easily reconstructed from these typical examples.

First suppose that the π-meson line runs between external points. Before we insert our photon with momentum q, the line looks like shown in Fig. 76. The three pion propagators

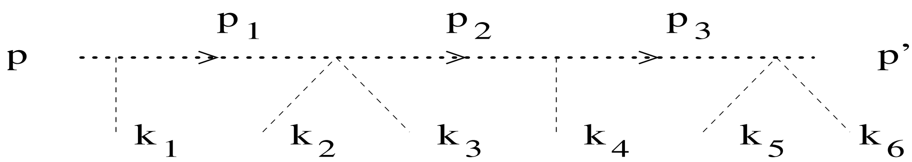


Figure 76. A line from the diagram for $G^{(N-1)}$.

have momenta $p_1 = p - k_1$, $p_2 = p - k_2 - k_3$, $p_3 = p - k_2 - k_3 - k_4$, and finally to $p' = p - k_2 - k_3 - k_4 - k_5 - k_6$. The corresponding expression had the form:

$$(p + p_1)_{\nu_1}\mathcal{G}_0(p_1)2g_{\nu_2\nu_3}\mathcal{G}_0(p_2)(p_2 + p_3)_{\nu_4}\mathcal{G}_0(p_3)2g_{\nu_5\nu_6} \tag{5.135}$$

We can insert our photon in 5 different places as shown in Fig. 77). The π-meson propa-

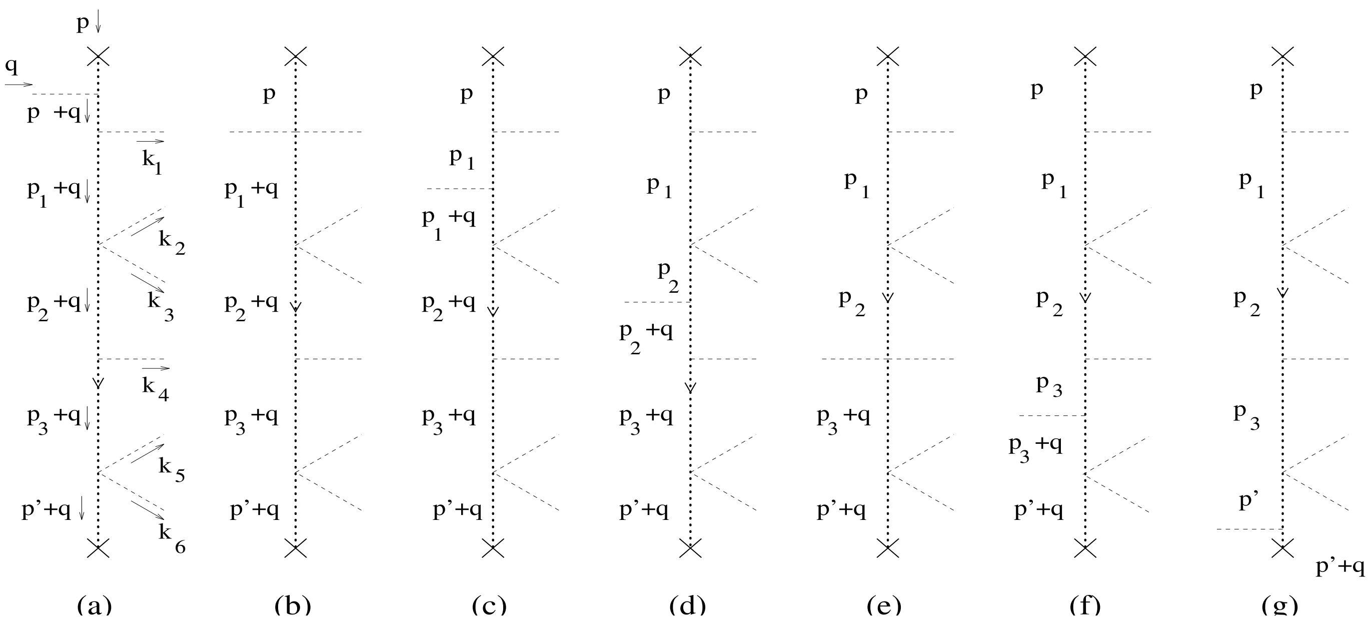


Figure 77. Possible insertions of the extra photon in the π-meson line shown in Fig. 55.

gators to the left of the new photon then have their momenta increased by q.

Let us multiply these diagrams by q_μ. The product of q_μ with the new $\pi\pi\gamma$ vertex can be written as:

$$q^\mu \Gamma_\mu(p_i+q, p_i) = q(2p_i+q) = (p_i+q)^2 - p_i^2 = \mathcal{G}^{-1}(p_i) - \mathcal{G}^{-1}(p_i+q)) \tag{5.136}$$

After multiplication by the adjacent π-meson propagators, we obtain [47]

$$\frac{1}{m^2-(p_i+q^2)} q(2p_i+q) \frac{1}{m^2-p_i^2} = \frac{1}{m^2-(p_i+q^2)} - \frac{1}{m^2-p_i^2} \tag{5.137}$$

If the corresponding vertex is at one of the ends of the line, there is only one attached propagator (recall that our line is truncated $\equiv$ multiplied by $(m^2-p^2)(m^2-(p')^2)$ since we consider $G^{\rm amp}$) so we get

$$\frac{1}{m^2-(p+q)^2} q(2p+q) = 1, \tag{5.138}$$

(recall that we consider the pions at the mass shell so $p^2 = (p')^2 = m^2$).

Alternatively, if we insert extra photon into the existing $\pi\pi\gamma$ vertex (making it the $\pi\pi\gamma\gamma$ vertex) the answer is simply

$$q^\mu 2g_{\mu\nu_i} = q_{\nu_i} \tag{5.139}$$

Using the above formulas, it is easy to get the answers for the diagrams shown in Fig. 77 in the following form:

$$\begin{aligned}
(a) &= -(p+p_1+2q)_{\nu_1}\mathcal{G}_0(p_1+q)2g_{\nu_2\nu_3}\mathcal{G}_0(p_2+q)(p_2+p_3+2q)_{\nu_4}\mathcal{G}_0(p_3+q)2g_{\nu_5\nu_6}\\
(b) &= 2q_{\nu_1}\mathcal{G}_0(p_1+q)2g_{\nu_2\nu_3}\mathcal{G}_0(p_2+q)(p_2+p_3+2q)_{\nu_4}\mathcal{G}_0(p_3+q)2g_{\nu_5\nu_6}\\
(c) &= (p+p_1)_{\nu_1}[\mathcal{G}_0(p_1+q)-\mathcal{G}_0(p_1)]2g_{\nu_2\nu_3}\mathcal{G}_0(p_2+q)(p_2+p_3+2q)_{\nu_4}\mathcal{G}_0(p_3+q)2g_{\nu_5\nu_6}\\
(d) &= (p+p_1)_{\nu_1}\mathcal{G}_0(p_1)2g_{\nu_2\nu_3}[\mathcal{G}_0(p_2+q)-\mathcal{G}_0(p_2)](p_2+p_3+2q)_{\nu_4}\mathcal{G}_0(p_3+q)2g_{\nu_5\nu_6}\\
(e) &= (p+p_1)_{\nu_1}\mathcal{G}_0(p_1)2g_{\nu_2\nu_3}\mathcal{G}_0(p_2)2q_{\nu_4}\mathcal{G}_0(p_3+q)2g_{\nu_5\nu_6}\\
(f) &= (p+p_1)_{\nu_1}\mathcal{G}_0(p_1)2g_{\nu_2\nu_3}\mathcal{G}_0(p_2)(p_2+p_3)_{\nu_4}[\mathcal{G}_0(p_3+q)-\mathcal{G}_0(p_3)]2g_{\nu_5\nu_6}\\
(g) &= (p+p_1)_{\nu_1}\mathcal{G}_0(p_1)2g_{\nu_2\nu_3}\mathcal{G}_0(p_2)(p_2+p_3)_{\nu_4}\mathcal{G}_0(p_3)2g_{\nu_5\nu_6}
\end{aligned} \tag{5.140}$$

By inspection, we see that the sum of all terms cancel. It is clear from the above analysis that this pattern of cancellations is general and we will have the casimilar cancellation for the π-meson line with arbtitrary number of photon emissions.

To complete the proof of Ward identity, we must consider the case when photon attaches to an internal π-meson loop. Before the insertion of the photon, a typical loop looks like shown in Fig. 78 The π-meson propagators have momenta p, $p_1 = p - k_1$, $p_2 = p - k_2 - k_3$, $p_3 = p - k_2 - k_3 - k_4$, and of course the sum of all photon momenta should vanish: $k_1 + k_2 + k_3 + k_4 + k_5 + k_6 = 0$. The loop integral then has the form:

$$\int \frac{d^4p}{(2\pi)^4 i}\mathcal{G}_0(p)(p+p_1)_{\nu_1}\mathcal{G}_0(p_1)2g_{\nu_2\nu_3}\mathcal{G}_0(p_2)(p_2+p_3)_{\nu_4}\mathcal{G}_0(p_3)2g_{\nu_5\nu_6} \tag{5.141}$$

[47] In this section, we will not write down the $i\epsilon$ factors in the denominators - instead, we will keep them always in mind.

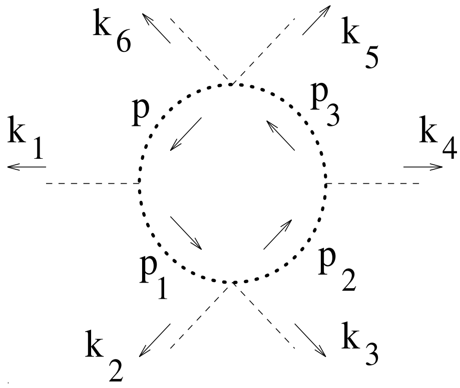


Figure 78. A typical π-meson loop.

Suppose now that we emit the photon with the momentum q between the vertices i and $i+1$. We now have an additional momentum q running around the loop because of the new vertex. By convention, this momentum enters together with k_1 and exits at the new vertex. The corresponding set of the diagrams is shown in Fig. 79

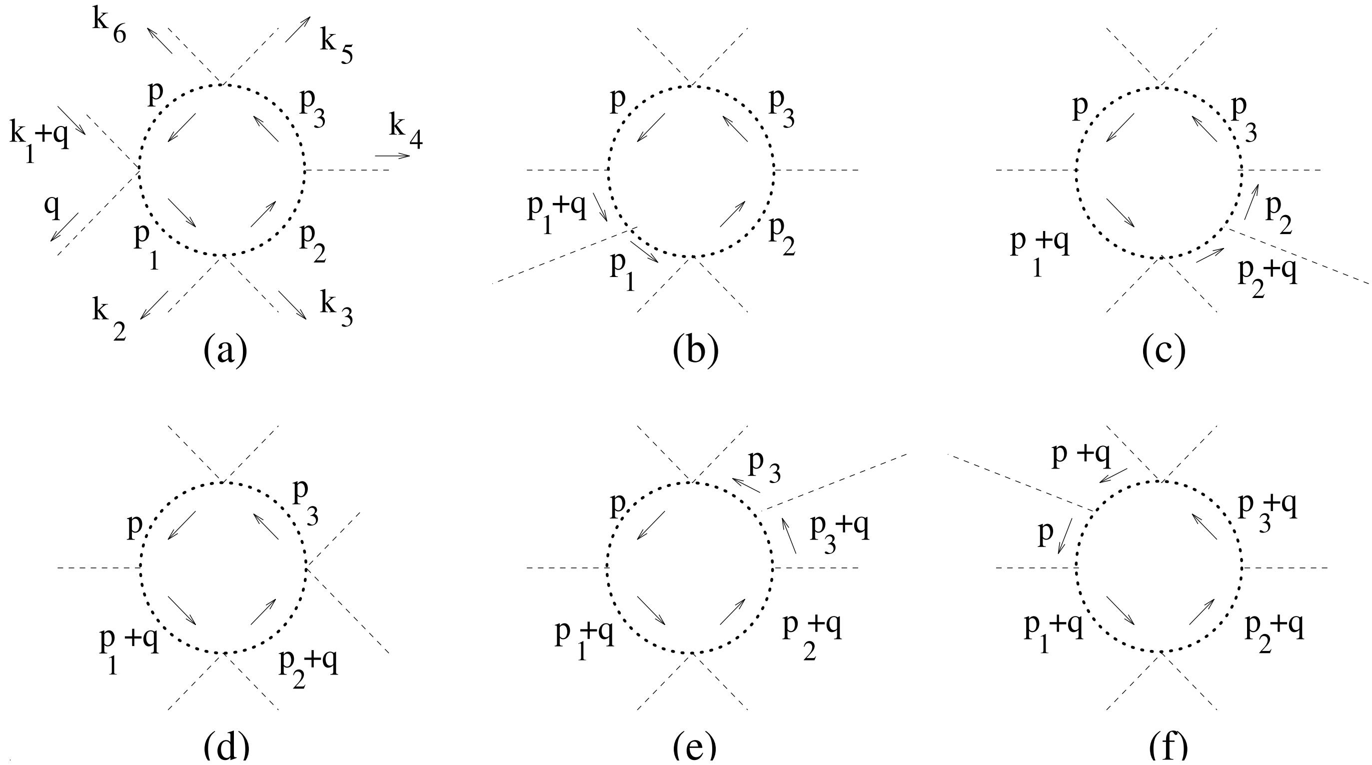


Figure 79. Possible insertions into the π-meson loop showh in Fig. 58 .

Using the above formulas, it is easy to get the answers for the diagrams shown in Fig.

77 in the following form:

$$\begin{aligned}
(a) &= 2q_{\nu_1}\mathcal{G}_0(p_1)2g_{\nu_2\nu_3}\mathcal{G}_0(p_2)(p_2+p_3)_{\nu_4}\mathcal{G}_0(p_3)2g_{\nu_5\nu_6}\mathcal{G}_0(p) \qquad (5.142)\\
(b) &= (p+p_1+q)_{\nu_1}[\mathcal{G}_0(p_1+q)-\mathcal{G}_0(p_1)]2g_{\nu_2\nu_3}\mathcal{G}_0(p_2)(p_2+p_3)_{\nu_4}\mathcal{G}_0(p_3)2g_{\nu_5\nu_6}\mathcal{G}_0(p)\\
(c) &= (p+p_1+q)_{\nu_1}\mathcal{G}_0(p_1+q)2g_{\nu_2\nu_3}[\mathcal{G}_0(p_2+q)-\mathcal{G}_0(p_2)](p_2+p_3)_{\nu_4}\mathcal{G}_0(p_3)2g_{\nu_5\nu_6}\mathcal{G}_0(p)\\
(d) &= (p+p_1+q)_{\nu_1}\mathcal{G}_0(p_1+q)2g_{\nu_2\nu_3}\mathcal{G}_0(p_2+q)2q_{\nu_4}\mathcal{G}_0(p_3)2g_{\nu_5\nu_6}\mathcal{G}_0(p)\\
(e) &= (p+p_1+q)_{\nu_1}\mathcal{G}_0(p_1+q)2g_{\nu_2\nu_3}\mathcal{G}_0(p_2+q)(p_2+p_3+2q)_{\nu_4}[\mathcal{G}_0(p_3+q)-\mathcal{G}_0(p_3)]2g_{\nu_5\nu_6}\mathcal{G}_0(p)\\
(g) &= (p+p_1+q)_{\nu_1}\mathcal{G}_0(p_1+q)2g_{\nu_2\nu_3}\mathcal{G}_0(p_2+q)(p_2+p_3+2q)_{\nu_4}\mathcal{G}_0(p_3+q)2g_{\nu_5\nu_6}[\mathcal{G}_0(p+q)-\mathcal{G}_0(p)]
\end{aligned}$$

It is easy to see that all the terms cancel, except the first, the second, and the last, so we obtain

$$\begin{aligned}
&\int \frac{d^4p}{(2\pi)^4 i}(p+p_1+q)_{\nu_1}\mathcal{G}_0(p_1+q)2g_{\nu_2\nu_3}\mathcal{G}_0(p_2+q)(p_2+p_3+2q)_{\nu_4}\mathcal{G}_0(p_3+q)2g_{\nu_5\nu_6}\mathcal{G}_0(p+q)-\\
&\quad - \int \frac{d^4p}{(2\pi)^4 i}(p+p_1-q)_{\nu_1}\mathcal{G}_0(p_1)2g_{\nu_2\nu_3}\mathcal{G}_0(p_2)(p_2+p_3)_{\nu_4}\mathcal{G}_0(p_3)2g_{\nu_5\nu_6}\mathcal{G}_0(p) \qquad (5.143)
\end{aligned}$$

Now, making shift $p \to p+q$ of the integration variable in the second term in r.h.s of eq.5.143 we obtain exactly the first term (with opposite sign) so the result is actually 0. Again, it is clear that this vanishing will be true any number of photon legs.

We are now ready to finish the proof. Suppose the amputated Green function G^N has n incoming π-meson lines and n outcoming. Let us consider the Green function G^{N-1} which lacks the photon with momentum q but is otherwise identical to G^N. To form $q^\mu G^N_\mu$ we must sum over all diagrams that contribute to G^{N-1}, and for each diagram, sum over all the points at which the photon could be inserted. Summing over insertion points along an internal loop in any diagram gives zero due to eqs. (5.142) and (5.143) and summing over all the insertions along the through-line in any diagram gives zero due to eq. (5.140) [48]. Thus, we have proved our Ward identity (5.134).

Now we will use this Ward identity in order to prove that we cannot create the unphysical longitudinal photons in the physical scattering processes. Indeed, the vector of polarization of the longitudinal photon is proportional to its 4-momentum, and therefore the product of this polarization vector and the corresponding amputated Green function (with all the particles on the mass shell) gives zero [49].

Second important consequence of Ward identity is that we can use any gauge for the photon propagator - the physical results will be the same. (Actually, I have already used this property when we were constructing Feynman propagator D^F and promised to prove it in the future, and now that future came into being). More accurately, if we use the trial propagator

$$\mathcal{D}^t_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} + k_\mu a_\nu + b_\mu k_\nu + c k_\mu k_\nu\right) \qquad (5.144)$$

instead of the Feynman one $\mathcal{D} = \frac{g_{\mu\nu}}{k^2}$ the elements of the S-matrix will remain the same. (Here a_ν, b_μ, and c can be arbitrary functions of k, not even nessesarily relativistic invariant)

[48] Of course, I mean here the generalization of these equation to the arbitrary number of photon legs, which can be easily done.

[49] It is sufficient that all the π-mesons are on the mass shell.

The proof actually repeats the above steps. Let us prove this statement by induction in number of internal photon lines I. First, consider I=1. This photon line can connect two through-going π-meson lines (or the photon may be attached to just one line), or one through-going line and one internal pion loop, or two loops (or both ends of the photon line may be attached to the same loop). Let us consider the term $a^\mu k_\mu$ in the propagator. In all of these cases, let us fix the attachment to the end corresponding to a^μ. Then we must first sum over all the attachments of the second tail (multiplied by k_μ) to the through-going line, or to the loop [50]. In both cases, we obtain zero as proved above (see eqs. (5.140) and (5.142)).

Now, let us proceed by induction. Suppose we have proved this statement (that we can use the propagator (5.144) instead of the Feynman one) for all the diagrams with I-1 internal photon lines. Let us take now an arbitrary diagram with I internal lines and let us remove an arbitrary line from this diagram. We will get then the diagram with I-1 internal lines (this diagram may turn out to be disconnected after such removal, but it will make no difference for us). Let us call that diagram G_{I-1}. Now, if we reinsert the removed photon in all possible ways, we will obtain a certain subset of the total set of diagrams with I internal photon lines. We will prove now that in this subset we can replace the photon propagator $\mathcal{D} = \frac{g_{\mu\nu}}{k^2}$ by (5.144).

First, note that due to induction proposal we can make this replacement of the propagator in all the photon lines except the reattached line. Next, let us take, say, the term $a^\mu k_\nu$ in this reattached propagator. We must sum over all the attachments of the two photon lines to the π-mesons. Let us at first sum over all the attachments of the end which goes multiplied by k_μ. Then, again, we have two possibilities, and the sum over all the insertions of this end into (a) through-going line and (b) internal line gives 0 after multiplication by k_ν. By induction, we have proved our property.

Part XXI

6 Relativistic particles with spin $\frac{1}{2}$

6.1 Non-relativistic spinors

The spin-$\frac{1}{2}$ particle such as electron can be described by two wave functions ψ^1 and ψ^2 which are amplitudes of probability to observe this particle with z-component of the spin $+\frac{1}{2}$ or $-\frac{1}{2}$. They can be assembled into one two-componennt wave function $\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$ The wavefunction of the spin-$\frac{1}{2}$ particle at rest is therefore

$$\Psi_0 = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} e^{-imt} \tag{6.1}$$

[50] After that, we must sum over all the attachmments of the first tail, but we will get the desired zero already at the first step.

In the case of spin-0 particle we have found the wavefuction of a moving particle - it has the form (3.26)

$$\Psi_p(x) = \frac{e^{-ipx}}{\sqrt{2p_0}} \tag{6.2}$$

Now we must find the wavefunction of the moving spin-$\frac{1}{2}$ particle . From general grounds, one should expect the two-component wavefunction of the same type:

$$\Psi_p(x) = \begin{pmatrix} \psi^1(p) \\ \psi^2(p) \end{pmatrix} \frac{e^{-ipx}}{\sqrt{2p_0}} \tag{6.3}$$

where the relativistic spinor $\psi^\alpha(p)$ is transformed under Lorentz transformations in a way compatible with the known physical properties of spin-$\frac{1}{2}$ particles (such as that the two of them can merge either in spin-0 or in spin-1 particle).

To warm up, let us recover in this way the transformation properties of the non-relativistic spinors. Suppose we make a rotation [51]

$$x_i \to R_{ij} x_j \tag{6.4}$$

The amplitude to discover the particle with spin up (or down) the new z' axis should be a linear combination of the corresponding amplitudes to discover spin parallel and antiparallel to the "old" z axis. Indeed, let us write down

$$\begin{aligned} |\mathrm{state}\rangle &= \xi^1|\uparrow_z\rangle + \xi^2|\downarrow_z\rangle \\ |\mathrm{rotated\ state}\rangle &= \xi^{1'}|\uparrow_{z'}\rangle + \xi^{2'}|\downarrow_{z'}\rangle \end{aligned} \tag{6.5}$$

Since $|\uparrow_{z'}\rangle = c_1|\uparrow_z\rangle + c_2|\uparrow_z\rangle$ (and similarly for $|\downarrow_{z'}\rangle$), the components of the spinor ξ' and η' in the new coordinates should be linear functions of the components in the old coordinates:

$$\xi^{1'} = a\xi^1 + b\xi^2, \qquad \xi^{2'} = c\xi^1 + d\xi^2 \tag{6.6}$$

where complex numbers a, b, c, d are functions of the parameters of the rotation R (6.4). They can be assembled to one 2×2 matrix

$$U(R) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6.7}$$

so the transformation of spinor is

$$\begin{pmatrix} (\xi')^1(p) \\ (\xi')^2(p) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi^1(p) \\ \xi^2(p) \end{pmatrix} \tag{6.8}$$

[51] Following the majority of the textbooks on quantum mechanics, we define here a matrix R fior the so-called active rotation. This means that when we write $x_i \to R_{ik}x_k$ we have in mind the following. Suppose we have a certain vector $\vec{r}$ with the coordinates x_i. The active rotation means that we grab this vector by hand and rotate it intil it fits into some new position specified by vector $\vec{r}'$ with the coordinates $x_i' = R_{ik}x_k$. For example, namely for such definition the matrix for the rotation on the angle ϕ around the z axis has the form (5.102). Similarly we define the active rotation R of the state: we take the state $|\Psi\rangle$ and rotate it in such a way that the nearby vector x would fit into $R_{ik}x_k$. Then this state undergoes transformation into some other state $|\Psi'\rangle = U(R)\Psi\rangle$. The wave function $\langle\vec{r}|\Psi\rangle$ is transformed then into $\langle\vec{r}|U(R)|\Psi\rangle$ so it appears that the arguments of the wavefunction are rotated by the $U^\dagger$ matrix in the opposite direction.

Let us now find the explicit form of these matrices using 3 pieces of the information:
(i) two spin-$\frac{1}{2}$ particles can form a spin-0 particles,
(ii) probability density to discover the particle in any of the spin states should be a scalar, and
(iii) two spin-$\frac{1}{2}$ particles can form a spin-1 state.

Let us start from the first property. Consider the two spinors ξ^α and η^β which we would like to merge in one spin-0 state. A general state of two spin-$\frac{1}{2}$ particles can be decomposed as follows

$$\begin{aligned}(\xi^1|\uparrow\rangle+\xi^2\downarrow\rangle)(\eta^1|\uparrow\rangle+\eta^2\downarrow\rangle) &= \frac{\xi^1\eta^2-\xi^2\eta^1}{\sqrt{2}}\frac{|\uparrow\rangle|\downarrow\rangle-\downarrow\rangle\uparrow\rangle}{\sqrt{2}} \\ +\,\xi^1\eta^1|\uparrow\rangle\uparrow\rangle+\frac{\xi^1\eta^2+\xi^2\eta^1}{\sqrt{2}}&\frac{|\uparrow\rangle|\downarrow\rangle+\downarrow\rangle\uparrow\rangle}{\sqrt{2}}+\xi^2\eta^2|\downarrow\rangle\downarrow\rangle\end{aligned} \tag{6.9}$$

The first term in the r.h.s. can be identified with the spin-0 combination of two spins $\frac{1}{2}$ and $-\frac{1}{2}$

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle\,|\downarrow\rangle-|\downarrow\rangle|\uparrow\rangle) \tag{6.10}$$

and therefore the “spinor” corresponding to the spin-0 state has the form

$$\xi^1\eta^2-\xi^2\eta^1 \tag{6.11}$$

It is easy to check that under the rotation (6.8) this combination transforms as

$$\xi^1\eta^2-\xi^2\eta^1\Rightarrow(ad-bc)(\xi^1\eta^2-\xi^2\eta^1) \tag{6.12}$$

which will be a scalar if our our U-matrix (6.7) has determinant 1. So, $\det U=1$ is the first property of matrix U which we established. Let us make here some technical refinements which we will need in the relativistic case. In order to give the convolution (6.11) a form similar to the scalar product of vectors let us introduce the spinors with lower (contravariant) indices:

$$\xi_1\stackrel{\text{def}}{\equiv}\xi^2,\quad \xi_2\stackrel{\text{def}}{\equiv}-\xi^1 \tag{6.13}$$

Then the spin-0 combination (6.11) is written as a "scalar product" of spinors:

$$\xi^\lambda\eta_\lambda=-\xi_\lambda\eta^\lambda \tag{6.14}$$

where $\lambda=1,2$ and the summation over the repeating indices is implied as usual. The relation between the spinors with upper and lower indices can be put in the form resembling the relation between covariant and contravariant components of the 4-vector $a^\mu=g^{\mu\nu}a_\nu$, $a_\mu=g^{\mu\nu}a^\nu$:

$$\xi^\lambda=\epsilon^{\lambda\rho}\xi_\rho\quad,\qquad \xi_\lambda=\epsilon_{\lambda\rho}\xi^\rho \tag{6.15}$$

where $\epsilon_{\alpha\beta}$ is the antisymmetric tensor in two dimensions. (From the eq. (6.13) we see that $\epsilon_{12}=-\epsilon_{21}=1$ and $\epsilon^{12}=-\epsilon^{21}=-1$ while $\epsilon^{11}=\epsilon^{22}=\epsilon_{11}=\epsilon_{22}=0$). Also, $\epsilon^{\alpha\lambda}\epsilon_{\lambda\beta}=\delta^\alpha_\beta$.

The second property of the matrices (6.7) follows from the fact that total probability density to discover our spin-$\frac{1}{2}$ particle in any of the states (with spin up or spin down)

$$\rho(x) = \rho_{up} + \rho_{down} = \xi^{1*}\xi^1 + \xi^{2*}\xi^2 \equiv \xi^\dagger\xi \tag{6.16}$$

should not change under rotations. Here we introduced the Hermitian conjugate spinor

$$\xi^\dagger = \left(\xi^{1*},\ \xi^{2*}\right) \tag{6.17}$$

which transforms under rotations with the help of the Hermitian conjugate matrix $U^\dagger$:

$$\xi \to U\xi \quad \Rightarrow \quad \xi^\dagger \to \xi^\dagger U^\dagger \tag{6.18}$$

Therefore, the requirement for the probability (6.16) to be a scalar is translated into the requirement of the unitarity of the matrix U:

$$\xi^\dagger\xi \to \xi^\dagger U^\dagger U\xi = \xi^\dagger\xi \quad \Rightarrow U^\dagger U = 1 \tag{6.19}$$

It is instructive to note that the complex conjugate spinors transform like the spinors with lower indices. This follows from the fact that both $\xi^\dagger\xi$ and $\xi_\alpha\eta^\alpha$ are scalars:

$$\left.\begin{array}{rl} \eta^\alpha & \to (U\eta)^\alpha \\ \xi_\alpha\eta^\alpha & - \text{ scalar} \end{array}\right\} \Rightarrow \xi_\alpha \to \left(\xi U^\dagger\right)_\alpha \tag{6.20}$$

$$\left.\begin{array}{rl} \xi^\alpha & \to (U\xi)^\alpha \\ \xi^{\dagger\alpha}\xi^\alpha & - \text{ scalar} \end{array}\right\} \Rightarrow \xi^{\dagger\alpha} \to \left(\xi^\dagger U^\dagger\right)^\alpha \tag{6.21}$$

Now, we have two conditions: that our matrix U is unitary and has determinant 1. Let us count the number of real parameters in this matrix. We have four complex numbers (a,b,c,d) which means 8 real parameters, but we have 2 conditions that $\det U = ad - bc = 1$ (both $\Re$ and $\Im$) and 3 more conditions follow from unitarity:

$$U^\dagger = U^{-1} \Rightarrow \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{6.22}$$

which means that we only have 3 independent real numbers which may be related to the three parameters defining the rotation. Now we shall fix these numbers using the requirement that two spin-$\frac{1}{2}$ particles can form the spin-1 particle.

The wavefunction of the massive vector ($\equiv$ spin-1) meson is described by the polarization vector which in the rest frame has the form (5.108):

$$\vec{e}^1 = (1,0,0), \quad \vec{e}^2 = (0,1,0), \quad \vec{e}^3 = (0,0,1) \tag{6.23}$$

(and time components of all the polarizaion vectors vanish in rest frame). As we saw above, the state with projection 1 on the spin on z axis corresponds to the circular polarization e^+ and the state with $s_z = -1$ - to e^-. (Similarly, it can be checked that projection 0 corresponds to e^3). Thus, the identification between spinors, states with projections of the spin on the z axis, and polarization vectors has the form [52]

$$\begin{array}{c} s_z = 1 \\ s_z = 0 \\ s_z = -1 \end{array} \Leftrightarrow \begin{array}{c} \xi^1\eta^1 \\ \frac{1}{\sqrt{2}}(\xi^1\eta^2 + \xi^2\eta^1) \\ \xi^2\eta^2 \end{array} \Leftrightarrow \begin{array}{c} |\uparrow\rangle\,|\uparrow\rangle \\ \frac{1}{\sqrt{2}}\,(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle) \\ |\downarrow\rangle|\downarrow\rangle \end{array} \Leftrightarrow \begin{array}{c} -\frac{|e^1\rangle + i|e^2\rangle}{\sqrt{2}} \\ |e^3\rangle \\ \frac{|e^1\rangle - i|e^2\rangle}{\sqrt{2}} \end{array} \tag{6.24}$$

Thus, the meson states corresponding to definite polarization vectors $\vec{e}^1, \vec{e}^2$, and $\vec{e}^3$ are:

$$\begin{aligned} |\vec{e}^1\rangle &= -\tfrac{1}{\sqrt{2}}(|s_z = 1\rangle - |s_z = -1\rangle) = -\frac{1}{\sqrt{2}}(|\uparrow\rangle|\uparrow\rangle - |\downarrow\rangle|\downarrow\rangle) \\ |\vec{e}^2\rangle &= \ \tfrac{i}{\sqrt{2}}(|s_z = 1\rangle + |s_z = -1\rangle) \ = \frac{i}{\sqrt{2}}(|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle) \\ |\vec{e}^3\rangle &= \qquad\quad \tfrac{1}{\sqrt{2}}|s_z = 0\rangle \qquad\quad = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle) \end{aligned} \tag{6.25}$$

and they should transform under (active) rotations as vectors [53]. We get

$$\vec{e}^{(k)}_i \to R_{ij}\vec{e}^{(k)}_j \quad \Leftrightarrow \quad |\vec{e}^{(k)}\rangle \to |\vec{e}^{(j)}\rangle R_{jk} \tag{6.26}$$

In principle, it is easy to figure out the transformation law of the spinor states directly from this formula, but it is more convenient to compare the transformation laws for the spinor and vector components of a certain spin-1 state described by symmetrical spinor $\xi^\alpha\eta^\beta$(symmetrical means that $\xi^\alpha\eta^\beta = \xi^\beta\eta^\alpha$ so the projection on spin 0 is absent):

$$\begin{gathered} (\xi^1|\uparrow\rangle + \xi^2|\downarrow\rangle)(\eta^1|\uparrow\rangle + \eta^2|\downarrow\rangle) = \\ \xi^1\eta^1|\uparrow\uparrow\rangle + \xi^2\eta^2|\downarrow\downarrow\rangle + \tfrac{\xi^1\eta^2+\xi^2\eta^1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = \\ \tfrac{1}{\sqrt{2}}(\xi^2\eta^2 - \xi^1\eta^1)|\vec{e}^1\rangle + \tfrac{-i}{\sqrt{2}}(\xi^1\eta^1 + \xi^2\eta^2)|\vec{e}^2\rangle + \tfrac{1}{\sqrt{2}}(\xi^1\eta^2 + \xi^2\eta^1)|\vec{e}^3\rangle \end{gathered} \tag{6.27}$$

Thus, the three combinations of spinor components

$$\begin{gathered} \frac{1}{\sqrt{2}}(\xi^2\eta^2 - \xi^1\eta^1) \\ \frac{-i}{\sqrt{2}}(\xi^1\eta^1 + \xi^2\eta^2) \\ \frac{1}{\sqrt{2}}(\xi^1\eta^2 + \xi^2\eta^1) \end{gathered} \tag{6.28}$$

should transform as a component of some 3-vector $\vec{a}$:

$$a_i \to R_{ik}a_k \tag{6.29}$$

[52] Actually, we can determine the relation between, say, $|\uparrow\rangle|\uparrow\rangle$ and $\frac{|e^1\rangle+i|e^2\rangle}{\sqrt{2}}$ only up to an arbitrary phase factor $e^{i\alpha}$. The particluar choice (6.24) correspond to the conventional set of Pauli matrices in eq. (6.31) below.

[53] In principle, one can prove that the states $-\frac{1}{\sqrt{2}}(|s_z = 1\rangle - |s_z = -1\rangle)$, $\frac{i}{\sqrt{2}}(|s_z = 1\rangle + |s_z = -1\rangle)$, and $\frac{1}{\sqrt{2}}|s_z = 0\rangle$ transform as a three components of a vector using the known transformation properties for the spin-one system, but we will use the physically motivated "proof" that the state of a vector meson labeled by a given polarization vector behaves under rotations as this polarization vector.

The combinations (6.28) become very nice if we use the spinors with lower indices. It is easy to see that

$$\begin{aligned}
\xi^2\eta^2 - \xi^1\eta^1 &= (\xi_1\ \xi_2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} \\
-i\xi^1\eta^1 - i\xi^2\eta^2 &= (\xi_1\ \xi_2)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} \\
\xi^1\eta^2 + \xi^2\eta^1 &= (\xi_1\ \xi_2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix}
\end{aligned} \tag{6.30}$$

which can be written down as

$$\xi_\alpha \vec{\sigma}^\alpha{}_\beta \eta^\beta \tag{6.31}$$

where σ_i are the usual Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.32}$$

So, we see that our spinors (ξ and η) should transform under rotations in such a way as to make the combination (6.31) to behave like a vector. In terms of the rotation matrix U this condition takes the form:

$$U^\dagger \sigma_i U = R_{ij}\sigma_j \tag{6.33}$$

The solution of this equation is well-known from group theory. If the rotation R is on the angle ϕ around the direction specified by unit vector $\vec{n}$ then

$$U(R) = e^{-\frac{i\phi}{2}\vec{\sigma}\cdot\vec{n}} \tag{6.34}$$

Let us verify it for the rotation around z axis. We must prove that

$$\begin{aligned}
e^{\frac{i\phi}{2}\sigma_z}\sigma_x e^{-\frac{i\phi}{2}\sigma_z} &= \sigma_x\cos\phi - \sigma_y\sin\phi \\
e^{\frac{i\phi}{2}\sigma_z}\sigma_y e^{-\frac{i\phi}{2}\sigma_z} &= \sigma_x\sin\phi + \sigma_y\cos\phi \\
e^{\frac{i\phi}{2}\sigma_z}\sigma_z e^{-\frac{i\phi}{2}\sigma_z} &= \sigma_z
\end{aligned} \tag{6.35}$$

(see the expression (5.102) for the matrix R for the rotation around the z axis). As to the third equation, it is trivial. Let us check the first one (the second is quite similar). First, let me remind the properties of Pauli matrices:

$$\sigma_i^2 = 1, \quad \sigma_x\sigma_y = i\sigma_z, \qquad \sigma_y\sigma_z = i\sigma_x, \qquad \sigma_z\sigma_x = i\sigma_y \tag{6.36}$$

We will need also the explicit form of the transformation matrix U(R) in r.h.s. of eq. (6.34). It can be easily otained using the properties (6.36):

$$e^{\frac{i\phi}{2}\vec{\sigma}\cdot\vec{n}} = I\cos\left(\frac{\phi}{2}\right) + i\vec{\sigma}\cdot\vec{n}\sin\left(\frac{\phi}{2}\right) \tag{6.37}$$

where I is the unit 2×2 matrix. Now, substituting eq. (6.37) in eq. (6.35) we obtain:

$$\begin{aligned}
&e^{\frac{i\phi}{2}\sigma_z}\sigma_x e^{-\frac{i\phi}{2}\sigma_z} = \\
&\left[I\cos\left(\tfrac{\phi}{2}\right) + i\sigma_z\sin\left(\tfrac{\phi}{2}\right)\right]\sigma_x\left[I\cos\left(\tfrac{\phi}{2}\right) - i\sigma_z\sin\left(\tfrac{\phi}{2}\right)\right]
\end{aligned} \tag{6.38}$$

Using the properties (6.36) this can be easily reduced to

$$\left[\sigma_x \cos^2\left(\frac{\theta}{2}\right) - 2\sigma_y \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \sigma_x \sin^2\left(\frac{\theta}{2}\right)\right] = \left[\sigma_x \cos\theta - \sigma_y \sin\theta\right] \tag{6.39}$$

Q.E.D. Similarly, the property (6.33) can be checked for other rotations. So, from the three requirements (i)-(iii) we have found the transformation properties of non-relativistic spinor - the spin part of the wavefunction of the non-relativistic particle. In the next Section we will do the same thing for the relaticvistic spinor - the spin part of the wavefunction of the relativistic particle.

Part XXII

6.2 Lorentz transformations

Let us at first recall the properties of Lorentz transformations. The Lorenz transformations are the linear transformations of the coordinates

$$x'_\mu = \Lambda_\mu^{\ \nu} x_\nu \tag{6.40}$$

which preserve the space-time interval

$$g_{\mu\nu} x'^{\mu} x'^{\nu} = g_{\mu\nu} x^\mu x^\nu \Rightarrow g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = g_{\alpha\beta} \tag{6.41}$$

If we will think about Λ^μ_α (and $g_{\mu\nu}$) as a matrix $[\Lambda]$ ($[g]$) with indices row=μ, column=ν, this property takes the form

$$[\Lambda^T][g][\Lambda] = [g] \tag{6.42}$$

Since $\det[\Lambda] = \det[\Lambda^T]$ we get $\det\Lambda = \pm 1$.

Among those Lorentz transformations are three rotations R and three Lorentz boosts L. We already know the matrices for the rotations. For example, the matrices corresponding to the rotation on angle ϕ around z, x, or y axis have the form (5.102):

$$R^z_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R^x_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}, \quad R^y_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 & 0 \\ 0 & \sin\phi & 0 & \cos\phi \end{pmatrix} \tag{6.43}$$

The arbitrary rotation (on the angle θ around arbitrary direction speccified by unit vector $\vec{n}$) can be represented as the product of three rotations (6.43). The three Lorentz boosts in z,x, and y directions are given by the matrices:

$$L^z_v = \begin{pmatrix} \cosh\phi & 0 & 0 & \sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix}, \quad L^x_v = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L^y_v = \begin{pmatrix} \cosh\phi & 0 & \sinh\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\phi & 0 & \cosh\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.44}$$

where tanh of the angle being the boost velocity: $\tanh\phi = v$. The Lorentz boost in arbitrary direction can be obtained by superposition of the boosts (6.44). Formally, the

Lorentz boost (for example, L^z) is the rotation on the complex angle -$i\phi$ in the corresponding plane (made of z and it). Note that both rotations and boosts have $\det\Lambda = 1$ [54]

Apart from those continuous Lorentz transformations there are so-called discrete Lorentz transformations – space reflection P: $\vec{r} \to -\vec{r}$ and time reflection T:$t \to -t$. The corresponding matrices are

$$\Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \Lambda_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.47}$$

These transformations are called discrete because you cannot obtain P or T by any superposition of continuous Lorentz transformations (6.43),(6.44). If is especially clear if we note that $\det\Lambda_P = \det\Lambda_T = -1$. Then if, say, P could have been obtained by a superposition of any number of rotations and boosts, its determinant would be 1 since the determinant of each rotation and boost is 1 (and the determinant of product of matrices is a product of determinants). So, to get P as a product of any number of rotations and boosts is impossible. In conclusion, note that the total reflection PT $x \to -x$ corresponding to the matrix

$$\Lambda_{PT} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.48}$$

with all -1 on the diagonal also cannot be obtained by any superposition of rotations and boosts (although its determinant is OK). In order to prove that, let us take the unit vector $n = (1, 0, 0, 0)$ and apply any number of rotations and boosts. It is clear from the formulas (6.43),(6.44) that neither rotation no boost can flip the sign of the first component of this

[54] To be precise, let us we define formally $X_1 = x^1$, $X_2 = x^2$, $X_3 = x^3$, and $X_4 = ix^0$ and forget for a second that X_4 is imaginary. Then we will obtain that the interval $x_0^2 - x_1^2 - x_2^2 - x_3^2$ which is conserved under Lorentz transformations can be rewritten as $x^2 = -(X_1^2 + X_2^2 + X_3^2 + X_4^2)$ so it is the usual $\vec{r}^2$ in the usual four-dimensional space up to a minus sign. (The name is for the usual space is "Euclidean space" in order to distinguish it from the space-time which is called "Minkowski space"). So, one should expect that the Lorentz transformations should be the rotations in these notations. Indeed, let us take the Lorentz boost in z direction (6.44) and rewite it in $X_1 \div X_4$ notations. It takes the form:

$$\begin{pmatrix} -iX_4 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} \to \begin{pmatrix} \cosh\phi & 0 & 0 & \sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} \begin{pmatrix} -iX_4 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} \tag{6.45}$$

which can be rewritten as

$$\begin{pmatrix} X_4 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} \to \begin{pmatrix} \cosh\phi & 0 & 0 & i\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} \begin{pmatrix} X_4 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \cos(i\phi) & 0 & 0 & \sin(i\phi) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin(i\phi) & 0 & 0 & \cos(i\phi) \end{pmatrix} \begin{pmatrix} X_4 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} \tag{6.46}$$

which is an usual rotation in the X_3, X_4 plane - only on imaginary angle $-i\phi$. So, the Lorentz boosts with the speed v in the x^3 (or x^1,or x^2) directions (6.44) are the rotations in the X_4, X_3 (or X_4, X_1, X_4, X_2) plane on the angle $\phi = -i\text{arctanh}v$

vector so $n_0' > 0$ after any number of rotations and boosts, and after PT the n_0 is -1, so PT cannot be obtainesd by continuous Lorentz transformations. It can be proved, however, that rotations + boosts + one of P, T, or PT give all the Lorentz transformations.

6.3 Relativistic spinors

Let us turn now to the relativistic spinors. Similarly to the non-relativistic case, we define the relativistic spinor as a pair of (complex) numbers which change under Lorentz transformations

$$x'_\mu = \Lambda_\mu^{\ \nu} x_\nu \tag{6.49}$$

in such a way that they can represent a wavefunction of the physical particle with spin $\frac{1}{2}$. First, from the superposition principle we conclude that the dependence of the new components of the wavefunction on the old ones is linear, so

$$\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \Rightarrow U(\Lambda) \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \tag{6.50}$$

In the non-relativistic case we had three requirements which fix the form of the 2×2 matrix U: (i) two spin-$\frac{1}{2}$ particles can form a spin-0 particle, (ii) two spin-$\frac{1}{2}$ particles can form a spin-1 particle, and (iii) the density $\xi^\dagger \xi$ should not change upon rotations. In the relativistic case, the condition (iii) is absent. Indeed, as we know from the theory of the relativistic scalar (and vector) particles, the density

$$\rho(t, \vec{r}) = \phi_+^*(t, \vec{r}) i \overset{\leftrightarrow}{\frac{d}{dt}} \phi_+(t, \vec{r}) \tag{6.51}$$

is a zero component of a four-vector

$$\rho^\mu(x) \overset{\text{def}}{\equiv} \phi_+^*(x) i \overset{\leftrightarrow}{\frac{d}{dx_\mu}} \phi_+(x) \tag{6.52}$$

rather than a scalar - and there is every reason to expect that the probability density to find the spinor particle (with any spin) will also transform like the time component of the vector rather than like a scalar (we will indeed see it later). It means that we no longer have a condition (6.19) that the matrix U is unitary. Still, the combination (6.11) of two spinors

$$\xi^\alpha \eta_\alpha = \xi^1 \eta^2 - \xi^2 \eta^1 \tag{6.53}$$

transforms via itself (see eq. (6.12) so it may be identified with the scalar particle made from the two spin-$\frac{1}{2}$ particles, provided the $(ad - bc) = \det U = 1$, and this gives us the first restriction on the matrix U. Actually, it is also the last restriction since we have 8-2=6 real parameters for the matrix U which can be fixed in a unique way by 6 parameters of Lorentz rotations. Now we will find the explicit form of the dependence of the elements of this U-matrix on the parameters of Lorentz transformation using the requrement that two spin-$\frac{1}{2}$ particles can merge in a vector particle with spin 1. Note that since the probability is not a scalar anymore, the complex conjugate spinor is no longer transformed as the spinor with lower indices (see eq. (6.20), (6.21)). In order to label the indices of the complex conjugate spinor we put the dot over the index, so

$$\xi^\alpha \to U^\alpha_\beta \xi^\beta \tag{6.54}$$

$$(\xi^*)^{\dot\alpha} \to (U^\dagger)_{\dot\beta}{}^{\dot\alpha}(\xi^*)^{\dot\beta} \tag{6.55}$$

Let me stress that the dot here has no extra meaning rather than being a label to remember that this index belongs to the spinor which transforms according to eq. (6.55) rather than the eq. (6.54). (I could have written these indices in a different color instead).

In order to guess the form of matrix U let us recall, that the Lorentz boosts look like rotations on the complex angle arctanh$v$. The crucial mathematical property for the vector to be formed from two spinors was the following property of the U matrices:

$$U^\dagger \sigma_i U = R_{ik}\sigma_k \tag{6.56}$$

(where $U = exp\frac{i\theta}{2}\vec\sigma\cdot\vec n$ for the rotation R on the angle θ around the direction $\vec n$). If so, the combination $\xi^\dagger\sigma_i\eta$ of two spinors ξ and η transforms like a vector under rotations:

$$(\xi^\dagger\sigma_i\eta) \to R_{ik}(\xi^\dagger\sigma_k\xi) \tag{6.57}$$

Let us write down this property for the rotation around z axis (see (6.35)):

$$\begin{aligned} U^\dagger\sigma_x U &= \cos\theta\sigma_x - \sin\theta\sigma_y \\ U^\dagger\sigma_y U &= \sin\theta\sigma_x + \cos\theta\sigma_y \end{aligned} \tag{6.58}$$

Now, since Lorentz boost in z direction looks like the rotation on the complex angle $i\theta$ we may try

$$U(i\theta) = e^{\frac{1}{2}\sigma_z\theta} \tag{6.59}$$

as a candidate for the U-matrix rotating the spinors under z-boost. Let us try. We must form a 4-vector from 2 spinors ξ and η and demonstrate that it is transformed in a proper way under Lorentz boost . An educated guess for this 4-vector formed by two spinors is

$$V^\mu = (V_0, \vec V) = (\xi^\dagger\eta, \xi^\dagger\vec\sigma\eta) \tag{6.60}$$

Indeed, it is easy to check that if we use our guess (6.59) for the U-matrix, we obtain for the boost in z-direction (cf. eq. (6.35):

$$\begin{aligned} V_0 = \xi^\dagger\eta &\to \xi^\dagger e^{\frac{1}{2}\sigma_z\theta}e^{\frac{1}{2}\sigma_z\theta}\eta = \xi^\dagger e^{\sigma_z\theta}\eta = \cosh\theta(\xi^\dagger\eta) + \sinh\theta(\xi^\dagger\sigma_z\eta) = V_0\cosh\theta + V_3\sinh\theta \\ V_3 = \xi^\dagger\sigma_z\eta &\to \xi^\dagger e^{\frac{1}{2}\sigma_z\theta}\sigma_z e^{\frac{1}{2}\sigma_z\theta}\eta = \xi^\dagger\sigma_z e^{\sigma_z\theta}\eta = \sinh\theta(\xi^\dagger\eta) + \cosh\theta(\xi^\dagger\sigma_z\eta) = V_0\sinh\theta + V_3\cosh\theta \\ V_1 = \xi^\dagger\sigma_x\eta &\to \xi^\dagger e^{\frac{1}{2}\sigma_z\theta}\sigma_x e^{\frac{1}{2}\sigma_z\theta}\eta = \xi^\dagger\left(\cosh\frac{\theta}{2} + \sigma_z\sinh\frac{\theta}{2}\right)\sigma_x\left(\cosh\frac{\theta}{2} + \sigma_z\sinh\frac{\theta}{2}\right)\eta = \xi^\dagger\sigma_x\eta = V_1 \end{aligned} \tag{6.61}$$

(and $V_2 \to V_2$ is obtained similarly to the the last line in the above equation). So, the four combinations (6.60) transform like components of the four-vector under the Lorentz boost in the z-direction. Similarly, one can check that this property will be true for the two remaining boosts in x and y directions. It is convenient to use the covariant notation $\sigma^\mu = (1, \vec{\sigma})$ then the vector formed by two spinors takes the nice form:

$$V^\mu = \xi^\dagger \sigma^\mu \eta \equiv (\xi^\dagger)^{\dot\alpha} (\sigma^\mu)_{\dot\alpha\beta} \eta^\beta \tag{6.62}$$

where we have displayed the spinor indices explicitly. So, the transformation of the spinor under the Lorentz boost with velocity v in the direction specified by vector n has the form:

$$\begin{aligned} \xi^\alpha &\to \left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)^\alpha_{\ \beta} \xi^\beta \\ (\xi^\dagger)^{\dot\alpha} &\to (\xi^\dagger)^{\dot\beta} \left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)_{\dot\beta}^{\ \dot\alpha} \end{aligned} \tag{6.63}$$

where $\theta = \text{arctanh}\, v$. We will need also the transformation law for the spinors with lower indices. It is easy to demonstrate that they are transformed in the opposite way:

$$\begin{aligned} \xi_\alpha &\to \xi_\beta \left(e^{-\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)^\beta_{\ \alpha} \\ \xi^\dagger_{\dot\alpha} &\to \xi^\dagger_{\dot\beta} \left(e^{-\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)^{\dot\beta}_{\ \dot\alpha} \end{aligned} \tag{6.64}$$

Indeed, let us take the same Lorentz boost in z direction as an example. We have:

$$\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \to \begin{pmatrix} e^{\frac{\theta}{2}} & 0 \\ 0 & e^{\frac{-\theta}{2}} \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \tag{6.65}$$

If we rewrite this transformation in terms of spinors with lower components we get:

$$\begin{pmatrix} -\xi_2 \\ \xi_1 \end{pmatrix} \to \begin{pmatrix} e^{\frac{\theta}{2}} & 0 \\ 0 & e^{\frac{-\theta}{2}} \end{pmatrix} \begin{pmatrix} -\xi_2 \\ \xi_1 \end{pmatrix} \tag{6.66}$$

Rearranging it in a proper way, we obtain:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \to \begin{pmatrix} e^{\frac{-\theta}{2}} & 0 \\ 0 & e^{\frac{\theta}{2}} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{6.67}$$

which corresponds to the transformation law (6.64). Similarly, the validity of eq. (6.64) can be checked for other Lorentz boosts.

The eq. (6.64) can be formally proved using the identity

$$\epsilon_{\alpha\beta}\epsilon^{\mu\nu} = -\delta^\mu_\alpha \delta^\nu_\beta + \delta^\nu_\alpha \delta^\mu_\beta \tag{6.68}$$

We get

$$\begin{aligned} &\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta \to \epsilon_{\alpha\beta}\left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)^\beta_{\ \gamma}\xi^\gamma = \epsilon_{\alpha\beta}\epsilon^{\gamma\lambda}\left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)^\beta_{\ \gamma}\xi_\lambda = (-\delta^\gamma_\alpha\delta^\lambda_\beta + \delta^\lambda_\alpha\delta^\gamma_\beta)\left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)^\beta_{\ \gamma}\xi_\lambda = \\ &\text{Tr}\left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)\xi_\alpha - \left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)^\beta_{\ \alpha}\xi_\beta = \\ &\left(2\cosh\frac{\theta}{2} - \big(\cosh\frac{\theta}{2} + \vec{\sigma}\cdot\vec{n}\sinh\frac{\theta}{2}\big)\right)^\beta_{\ \alpha}\xi_\beta = \left(e^{-\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)^\beta_{\ \alpha}\xi_\beta \end{aligned} \tag{6.69}$$

One may also define the matrices $\left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)$ with two upper or two lower indices using the antisymmetric tensor $\epsilon^{\alpha\beta}$ to raise the index and $\epsilon_{\alpha\beta}$ to lower the index. For example, by lowering the index in the first eq. (6.63) one obtains

$$
\begin{aligned}
?\xi_\alpha \to \left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)_{\alpha\beta}\xi^\beta = \\
\left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)_\alpha^{\ \gamma}\epsilon_{\gamma\beta}\xi^\beta = \left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)_\alpha^{\ \gamma}\xi_\gamma
\end{aligned}
\tag{6.70}
$$

It is easy to see that

$$
\left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)_\alpha^{\ \beta} = ?\epsilon_{\alpha\gamma}\epsilon^{\beta\lambda}\left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)^\gamma_{\ \lambda} = \left(e^{-\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)^\beta_{\ \alpha}
\tag{6.71}
$$

in agreement with eq. (6.64).

Part XXIII

6.4 Neutrinos and Weyl equation

In the previous Section we learned that there are two types of relativistic spinors which can be written down with covariant or contravariant indices : ξ^α or ξ_α and $\eta^{\dot\alpha}$ or $\xi_{\dot\alpha}$. (Mathematically, they correpspond to two different representation of Lorentz group). Under the rotations on the angle ϕ around the $\vec{n}$ axis, they transform as follows:

$$
\begin{aligned}
&\xi^\alpha \to \left(e^{-i\frac{1}{2}\vec{\sigma}\cdot\vec{n}\phi}\right)^\alpha_{\ \beta}\xi^\beta \qquad && \xi_\alpha \to \xi_\beta\left(e^{i\frac{1}{2}\vec{\sigma}\cdot\vec{n}\phi}\right)^\beta_{\ \alpha} \\
&\eta^{\dot\alpha} \to \eta^{\dot\beta}\left(e^{i\frac{1}{2}\vec{\sigma}\cdot\vec{n}\phi}\right)_{\dot\beta}^{\ \dot\alpha} && \eta_{\dot\alpha} \to \left(e^{-i\frac{1}{2}\vec{\sigma}\cdot\vec{n}\phi}\right)_{\dot\alpha}^{\ \dot\beta}\eta_{\dot\beta}
\end{aligned}
\tag{6.72}
$$

Similarly, under Lorentz boosts in the direction specified by $\vec{n}$ and the velocity corresponding to θ (arctanh$\theta = v$) they transform in a following way:

$$
\begin{aligned}
&\xi^\alpha \to \left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)^\alpha_{\ \beta}\xi^\beta \qquad && \xi_\alpha \to \xi_\beta\left(e^{-\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)^\beta_{\ \alpha} \\
&\eta^{\dot\alpha} \to \eta^{\dot\beta}\left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)_{\dot\beta}^{\ \dot\alpha} && \eta_{\dot\alpha} \to \left(e^{-\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)_{\dot\alpha}^{\ \dot\beta}\eta_{\dot\beta}
\end{aligned}
\tag{6.73}
$$

The complex conjugate spinors are transformed according to the "mnemonic rule"

$$
(\xi^\alpha)^\dagger = (\xi^\dagger)^{\dot\alpha}, \qquad (\xi_\alpha)^\dagger = (\xi^\dagger)_{\dot\alpha}, \qquad (\eta^{\dot\alpha})^\dagger = (\eta^\dagger)^\alpha, \qquad (\eta_{\dot\alpha})^\dagger = (\eta^\dagger)_\alpha, \tag{6.74}
$$

as can be easily seen from the complex conjugation of formulas (6.72) and (6.73). For example,

$$
\begin{aligned}
\xi^\alpha \overset{\text{rotation}}{\to} \left(e^{-i\frac{1}{2}\vec{\sigma}\cdot\vec{n}\phi}\right)^\alpha_{\ \beta}\xi^\beta \quad &\Rightarrow \quad (\xi^\alpha)^* \to \left(e^{i\frac{1}{2}\vec{\sigma}\cdot\vec{n}\phi}\right)_\beta^{\ \alpha}(\xi^\beta)^* \\
\xi^\alpha \overset{\text{boost}}{\to} \left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)^\alpha_{\ \beta}\xi^\beta \quad &\Rightarrow \quad (\xi^\alpha)^* \to \left(e^{\frac{1}{2}\vec{\sigma}\cdot\vec{n}\theta}\right)_\beta^{\ \alpha}(\xi^\beta)^*
\end{aligned}
\tag{6.75}
$$

which gives to the first mnemonic rule in Eq. (6.74) since the behavior of $(\xi^\alpha)^*$ under rotations and boost corresponds to that of $\eta^{\dot\alpha}$ as can be seen from Eqs. (6.72) and (6.73).

In the non-relativistic theory, the wavefunction of the paricle is described by the non-relativistic spinor which transforms under rotations according to eq. (6.34). There are two relativistic spinors of that sort: ξ^α and $\eta_{\dot\alpha}$. The question is which of these spinors can be taken as a wavefunction for the relativistic spin-$\frac{1}{2}$ particle. Let us take the first of them ξ^α and make a try.

The Lorentz transformation of spinor κ^α is

$$\kappa^\alpha \ \rightarrow \ \left(e^{\frac{1}{2}\vec\sigma\cdot\vec n\theta}\right)^\alpha_{\ \beta}\kappa^\beta = \left[\cosh\left(\frac{\theta}{2}\right) + \vec\sigma\cdot\vec n\sinh\left(\frac{\theta}{2}\right)\right]^\alpha_{\ \beta}\kappa^\beta \tag{6.76}$$

so our candidate for the wavefunction of a spinor particle moving with velocity v in the direction specified by vector $\vec n$ has the form:

$$(\phi_p^+)^\alpha(x) = \frac{1}{\sqrt{2m}\sqrt{p_0+m}}(p_0+m+|\vec p|\vec\sigma\cdot\vec n)^\alpha_{\ \beta}\kappa^\beta\frac{e^{-ip_0(t-vx_3)}}{\sqrt{2p_0}} \tag{6.77}$$

where κ^β is a certain spinor in the rest frame ($\begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$ (or any linear combination of these spinors) and. In the derivation of the above formula we used the equations $\cosh\frac{\theta}{2} = \sqrt{\frac{p_0+m}{2m}}$ and $\sinh\frac{\theta}{2} = \frac{|\vec p|}{\sqrt{(p_0+m)2m}}$ which follow from $\cosh\theta = \frac{p_0}{m}$, $\sinh\theta = \frac{|\vec p|}{m}$. (Here $p^\mu = (p_0, mv\vec n)$ is the 4-momentum of the particle). It is convenient to rewrite Eq. (6.77) in 4-dim notations using the set of four σ-matrices defined as (cf. (6.62))

$$(\sigma^\mu)_{\dot\alpha\alpha} \ = \ (1,\vec\sigma), \qquad\qquad (\bar\sigma^\mu)^{\alpha\dot\alpha} \ = \ (1,-\vec\sigma) \tag{6.78}$$

The matrices of transformation under the Lorentz boost (6.73) with velocity v in the direction specified by vector $\vec n$ take the form ($p^\mu = (p_0, mv\vec n)$)

$$\begin{aligned}
(e^{\frac{1}{2}\vec\sigma\cdot\vec n\theta})^\alpha_{\ \beta} &= \frac{\left(m+\bar\sigma_\mu p^\mu\sigma_0\right)^\alpha_{\ \beta}}{\sqrt{2m(p_0+m)}}, \qquad \left(e^{\frac{1}{2}\vec\sigma\cdot\vec n\theta}\right)_{\dot\beta}^{\ \dot\alpha} = \frac{\left(m+\sigma_0\bar\sigma_\mu p^\mu\right)_{\dot\beta}^{\ \dot\alpha}}{\sqrt{2m(p_0+m)}} \\
(e^{-\frac{1}{2}\vec\sigma\cdot\vec n\theta})^\beta_{\ \alpha} &= \frac{\left(m+\bar\sigma_0\sigma_\mu p^\mu\right)^\beta_{\ \alpha}}{\sqrt{2m(p_0+m)}}, \qquad \left(e^{-\frac{1}{2}\vec\sigma\cdot\vec n\theta}\right)_{\dot\beta}^{\ \dot\alpha} = \frac{\left(m+\sigma_\mu\bar\sigma_0 p^\mu\right)_{\dot\beta}^{\ \dot\alpha}}{\sqrt{2m(p_0+m)}}
\end{aligned} \tag{6.79}$$

Here the unit matrices $(\sigma_0)_{\dot\alpha\alpha}$ and $(\bar\sigma^\mu)^{\alpha\dot\alpha}$ were introduced to match dotted and undotted indices in Eq. (6.79) to those of Eq. (6.73)).

In these notations the wavefunction (6.77) reads

$$\frac{(m+p_\mu\bar\sigma^\mu\sigma_0)^\alpha_{\ \beta}}{\sqrt{2m(p_0+m)}}\kappa^\beta\frac{e^{-ip_0(t-\vec v\cdot\vec x)}}{\sqrt{2p_0}} \tag{6.80}$$

It is convenient to change slightly the normalization of the spinors - to multiply them by $\sqrt{2m}$ so the final form of the wavefunction (6.4.3) is

$$\phi_p^{+\alpha}(x) = \frac{(m+p\cdot\bar\sigma\sigma_0)^\alpha_{\ \beta}}{\sqrt{2(p_0+m)}}\kappa^\beta\frac{1}{\sqrt{2p_0}}e^{-ip_0(t-\vec v\cdot\vec x)} \tag{6.81}$$

Let us discuss now the behaviour of this candidate for wavefunction under spatial inversion (parity transformation) P (see eq. (6.47). Consider, for example, the spin-$\frac{1}{2}$ particle with positive helicity moving with speed v in the $+z$ direction. The corresponding wavefunction is (second(+) is the helicity):

$$\phi_p^{+(+)}(x) = \frac{p_0}{\sqrt{2(p_0+m)}}(1+\frac{m}{p_0}+v\sigma_z)\begin{pmatrix}1\\0\end{pmatrix}\frac{1}{\sqrt{2p_0}}e^{-ip_0(t-vx_3)} \tag{6.82}$$

We want to find the transformation of the wavefunction (6.82) under spatial reflection. First, let us note that if we look in a mirror on a particle moving in the z direction and spinning in counterclockwise we will see a particle moving in the opposite direction but spinning in the same way (see Fig. 80)

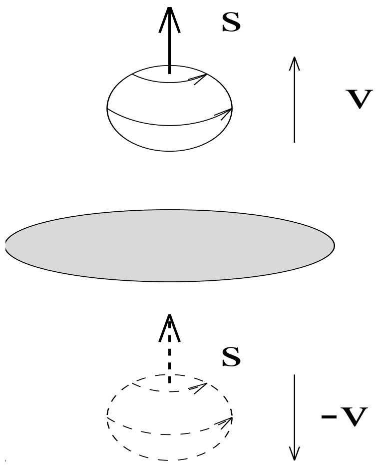


Figure 80. Reflection of a spinning particle.

Therefore, the wavefunction of a reflected positive-helicity particle takes the form [55]:

$$\frac{p_0}{\sqrt{2(p_0+m)}}(1+\frac{m}{p_0}+v\sigma_z)\begin{pmatrix}1\\0\end{pmatrix}\frac{1}{\sqrt{2p_0}}e^{-ip_0(t+vx_3)} \tag{6.83}$$

and if we want to have a mirrored paricle moving still in positive z direction it corresponds to $-v$ in the above formula, so the wavefunction of the reflected particle moving with speed v in positive z direction is:

$$\chi_p^{+(+)}(x) = \frac{p_0}{\sqrt{2(p_0+m)}}(1+\frac{m}{p_0}-v\sigma_z)\begin{pmatrix}1\\0\end{pmatrix}\frac{1}{\sqrt{2p_0}}e^{-ip_0(t-vx_3)} \tag{6.84}$$

The important point is that the wavefuctions (6.84) and (6.82) are different spinors. Formally, they transform in a different way under Lorentz boosts. Indeed, in the rest frame

[55] One may ask a question why we did not make a substitution $v \leftrightarrow -v$ in the spin part of the wavefunction. The short answer is that we believe the the quantum analog of the classical property that the spin does not change is that you do not alter the spin part of the wavefunction. A more refined argument is given below. We know that the probability density $\sim \xi^\dagger\xi \sim (1,0)\big(p_0+m+vp_0\sigma_z\big)\big(p_0+m+vp_0\sigma_z\big)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ transforms like a vector. Therefore, its time component should not change under spacial reflection which selects the form (6.83) since the alternative form (with $-v$ instead of v in the spin part) *does* change $\xi^\dagger\xi$.

the spinors (6.82) and (6.84) coincide, but after the Lorentz boost with velocity v in $+z$ direction they become different:

$$\left(\phi_p^{+(+)}\right) \to \frac{p_0}{\sqrt{2(p_0+m)}}(1+\frac{m}{p_0}+v\sigma_z)\left(\phi_p^{+(+)}\right) = e^{\frac{1}{2}\theta\sigma_z}\left(\phi_p^{+(+)}\right) \tag{6.85}$$

$$\left(\chi_p^{+(+)}\right) \to \frac{p_0}{\sqrt{2(p_0+m)}}(1+\frac{m}{p_0}-v\sigma_z)\left(\chi_p^{+(+)}\right) = e^{-\frac{1}{2}\theta\sigma_z}\left(\chi_p^{+(+)}\right) \tag{6.86}$$

The eqs. (6.85) and (6.86) correspond to the different transformation laws, (6.63) and (6.64). So, the parity reflection of a certain spinor ξ^α transforming according to (6.63) is a "dotted" spinor $\eta_{\dot\alpha}$ which transforms according to (6.64).

In the case of negative-helicity particle with the wavefunction

$$\phi_p^{+(-)}(x) = \frac{p_0}{\sqrt{2(p_0+m)}}(1+\frac{m}{p_0}+v\sigma_z)\begin{pmatrix}0\\1\end{pmatrix}\frac{1}{\sqrt{2p_0}}e^{-ip_0(t-vx_3)} \tag{6.87}$$

the corresponding wavefunction of the reflected particle is:

$$\chi_p^{+(-)}(x) = \frac{p_0}{\sqrt{2(p_0+m)}}(1+\frac{m}{p_0}-v\sigma_z)\begin{pmatrix}0\\1\end{pmatrix}\frac{1}{\sqrt{2p_0}}e^{-ip_0(t-vx_3)} \tag{6.88}$$

The most explicit way to demonstrate the difference between the wavefunctions (6.82) and (6.84) (or between (6.87) and (6.89) is to consider a massless spin-$\frac{1}{2}$ particle - neutrino.

In the case of massless particle $v=1$ and only two of the wavefunctions (6.82), (6.84) - (6.89) survive - the other two vanish. So, we have two possible wavefunctions coresponding to massless spin-$\frac{1}{2}$ particle moving in $+z$ direction:

$$\begin{aligned}\phi_p^{+(+)}(x) &= \frac{1}{2}(1+\sigma_z)\begin{pmatrix}1\\0\end{pmatrix}e^{-ip_0(t-x_3)} &=& \begin{pmatrix}1\\0\end{pmatrix}e^{-ip_0(t-x_3)}\\ \chi_p^{+(-)}(x) &= \frac{1}{2}(1-\sigma_z)\begin{pmatrix}0\\1\end{pmatrix}e^{-ip_0(t-x_3)} &=& \begin{pmatrix}0\\1\end{pmatrix}e^{-ip_0(t-x_3)}\end{aligned} \tag{6.89}$$

The first wavefunction $\phi_p^{+(+)}(x)$ describes a particle with positive helicity – antineutrino ("anti" stands for historical reasons). The second function $\chi_p^{+(-)}(x)$ can be identified with wavefunction of the negative-helicity neutrino. So, if we look in a mirror on a neutrino moving on us we will see an antineutrino running from us. The existence of massless spin-$\frac{1}{2}$ particles means that, in general, the parity is not an exact symmetry of the Nature.

The wavefunction of antineutrino which moves in the direction specified by polar angle θ and asimuthal angle ϕ can be obtained by the rotation of the wavefunction (6.89) on the angle θ around the axis $\hat n = -\vec e_1\sin\phi + \vec e_2\cos\phi$ (see Fig (81)) [56].

The matrix of rotation (6.34) has the form:

$$U(R) = e^{-\frac{i\phi}{2}\vec\sigma\cdot\vec n} \;=\; \cos(\theta/2)+\sin(\theta/2)(i\sigma_x\sin\phi - i\sigma_y\cos\phi) = \begin{pmatrix}\cos(\theta/2) & -e^{-i\phi}\sin(\theta/2)\\ e^{i\phi}\sin(\theta/2) & \cos(\theta/2)\end{pmatrix} \tag{6.90}$$

[56] This rotation is not unique one can rotate $\vec e_3$ to $\frac{\vec p}{|\vec p|}$ in an infinite number of ways. But all of them are equivalent since the corresponding wavefunctions differ only by the trivial phase factor, see eq. (8.34) in the Appendix.

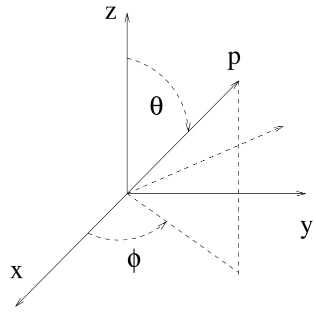


Figure 81. Rotation to the arbitrary $\vec{p}$.

so the wavefunction of the antineutrino is

$$\left(\phi_p^{+(+)}\right)^\alpha(x) = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} e^{-ipx} \tag{6.91}$$

The corresponding expression for the wavefunction of a neutrino has the form

$$\left(\chi_p^{+(-)}\right)_{\dot\alpha}(x) = \begin{pmatrix} \cos(\theta/2) & -e^{-i\phi}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ipx} = \begin{pmatrix} -e^{-i\phi}\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} e^{-ipx} \tag{6.92}$$

The "dotted" index of the spinor (6.92) corresponds to the transformation law of the P-inversed spinor. Similarly to the case of charged π-mesons, we will try to describe neutrino and antineutrino by one complex spinor field ν. As we know from the example of π-mesons in such type of description the positive-frequency part of the field can be identified with the wavefunction of the particle (antineutrino) and positive–frequency part of complex conjugate field ν^* with wavefunction of the antiparticle (neutrino). Therefore, the classical neutrino field can be represented as

$$\nu^\alpha = \int \frac{d^3p}{(2\pi)^3}\left(c(p)(\phi_p^{+(+)})^\alpha e^{-ipx} + d^*(p)\epsilon^{\alpha\beta}(\chi_p^{-(-)})_\beta e^{ipx}\right) \tag{6.93}$$

and similar expression for complex conjugate field is

$$(\nu^*)_{\dot\alpha} = \int \frac{d^3p}{(2\pi)^3}\left(d(p)(\chi_p^{+(-)})_{\dot\alpha})e^{-ipx} + \epsilon_{\dot\alpha\dot\beta}c^*(p)(\phi_p^{-(+)})^{\dot\beta}e^{ipx}\right) \tag{6.94}$$

where we have lowered the index $\dot\alpha$ in order to have the neutrino wavefunction (6.73) as the positive -frequency part of our field. Note that $(\chi_p^{+(-)})^* = \chi_p^{-(-)}$ and $(\phi_p^{+(+)})^* = \phi_p^{-(+)}$.

This classical neutrino field satisfies so-called Weyl equation:

$$\sigma^\mu_{\dot\alpha\alpha}\frac{\partial}{\partial x^\mu}\nu^\alpha(x) = 0 \tag{6.95}$$

where $\sigma^\mu = (1, \vec\sigma)$ as defined in previous Lecture.

Let us prove this. First, since

$$\bar\sigma\cdot p \;=\; |\vec p|\begin{pmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{pmatrix}, \qquad \sigma\cdot p \;=\; |\vec p|\begin{pmatrix} 1-\cos\theta & -e^{-i\phi}\sin\theta \\ -e^{i\phi}\sin\theta & 1+\cos\theta \end{pmatrix} \tag{6.96}$$

we get from Eqs. (6.90) and (6.91)

$$(\phi_p^{+(+)})^\alpha = [(\bar\sigma\cdot p)\sigma_0]^\alpha{}_\beta\varphi^\beta e^{-ipx} \qquad \text{and} \qquad (\chi_p^{+(-)})_{\dot\beta} = [(\sigma\cdot p)\bar\sigma_0]_{\dot\beta}{}^{\dot\gamma}\kappa_{\dot\gamma}e^{-ipx} \tag{6.97}$$

where

$$\varphi^\beta \equiv \frac{1}{2|\vec p|\cos(\theta/2)}\begin{pmatrix}1\\0\end{pmatrix} \qquad \text{and} \qquad \kappa_{\dot\gamma} \equiv \frac{1}{2|\vec p|\cos(\theta/2)}\begin{pmatrix}0\\1\end{pmatrix} \tag{6.98}$$

We need $(\chi_p^{-(-)})_\beta = \left[(\chi_p^{+(-)})_\beta\right]^*$ so from the mnemonic rule (6.74) we obtain

$$(\chi_p)_{\dot\beta} = [(p{\cdot}\sigma)\bar\sigma_0]_{\dot\beta}{}^{\dot\gamma}\kappa_{\dot\gamma}e^{-ipx} = (p_0-\vec\sigma{\cdot}\vec p)_{\dot\beta}{}^{\dot\gamma}\kappa_{\dot\gamma}e^{-ipx} \Rightarrow (\chi_p^*)_\beta = \kappa_\gamma(p_0-\vec\sigma{\cdot}\vec p)^\gamma{}_\beta e^{ipx} = \kappa_\gamma[\bar\sigma_0(p{\cdot}\sigma)]^\gamma{}_\beta e^{ipx}$$

since κ is real. Now, using the formula $\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\dot\beta\beta} = (\bar\sigma^\mu)^{\alpha\dot\alpha}$ we obtain $\epsilon^{\alpha\beta}(\sigma_\mu)_{\dot\alpha\beta} = \epsilon_{\dot\alpha\dot\beta}(\bar\sigma_\mu)^{\alpha\dot\beta}$ and therefore

$$\begin{aligned}\epsilon^{\alpha\beta}(\chi_p^*)_\beta &= \kappa_\gamma\epsilon^{\alpha\beta}(\bar\sigma_0)^{\gamma\dot\alpha}(p\cdot\sigma)_{\dot\alpha\beta}e^{ipx}\\ &= \kappa_\gamma\epsilon_{\dot\alpha\dot\beta}(\bar\sigma_0)^{\gamma\dot\alpha}(p\cdot\bar\sigma)^{\alpha\dot\beta}e^{ipx} = \kappa_\gamma\epsilon^{\gamma\beta}(\sigma_0)_{\dot\beta\beta}(p\cdot\bar\sigma)^{\alpha\dot\beta}e^{ipx} = [(p\cdot\bar\sigma)\sigma_0]^\alpha{}_\beta\kappa^\beta e^{ipx}\end{aligned} \tag{6.99}$$

Thus, we get

$$\nu^\alpha = \int\frac{d^3p}{(2\pi)^3}\left(c(p)[(p\cdot\bar\sigma)\sigma_0]^\alpha{}_\beta\varphi^\beta e^{-ipx} + d^*(p)[(p\cdot\bar\sigma)\sigma_0]^\alpha{}_\beta\kappa^\beta e^{ipx}\right) \tag{6.100}$$

and the equation (6.93) immediately follows from the fact that in the momentum representation

$$p_\mu\sigma^\mu_{\dot\alpha\alpha}p^\nu(\bar\sigma^\nu)^{\alpha\dot\beta} = p^2 = 0 \tag{6.101}$$

so if we apply the operator $p_\mu\sigma^\mu_{\dot\alpha\alpha}$ to the r.h.s. of eq. (6.93) we get 0. (Recall that $(\sigma^\mu)_{\dot\alpha\beta}\bar\sigma^\nu)^{\beta\dot\beta} + \mu\leftrightarrow\nu = 2g_{\mu\nu}\delta^{\dot\beta}_{\dot\alpha}$). This equation was proposed by Weyl in 1929, but was rejected at that time since it leads to the parity violation.

It should be mentioned that in this case our logic was anti-historical: we started from guessing of the wavefunctions of neutrinos and then derived the classical equations for the neutrino field. This is because we cannot observe the classical neutrino field by our eyes (or simple devices). If we could, then the Weyl equation would probably be called fifth Maxwell equation or smth of that sort and would be well studied in classical physics. In this case, we could proceed in a "normal" way: expand the solution of this equation into plane waves (6.93), (6.94) and make a wild guess that the positive-frequency part of r.h.s. of eq. (6.93) has a probabilistic interpretation so it can serve as the wavefunction of a particle "newtrino" which is a quantum of our field (and positive-frequency part of ξ^* can be interpreted as a wavefunction of a free "antinewtrino").

Part XXIV

6.5 Bispinors and Dirac equation

Let us return now to the discussion of spin-$\frac{1}{2}$ particles with mass. If we boost the spinor ξ^α which in the rest frame had a form

$$\begin{pmatrix}1\\0\end{pmatrix}e^{-imt}/\sqrt{2p_0}$$

we will get the expression (6.81):

$$\xi^\alpha = \frac{1}{\sqrt{2(p_0+m)}}(m+p_0+\vec{p}\cdot\vec{\sigma})\begin{pmatrix}1\\0\end{pmatrix}\frac{1}{\sqrt{2p_0}}e^{-ipx} \tag{6.102}$$

where $\vec{v}$ specifies the direction of the boost. As we discussed in the previous section, the spinor corresponding to the reflected particle (moving in the same direction) can be obtained in two steps: (**i**) prepare the spinor corresponding to the particle boosted in the opposite direction

$$\xi^\alpha(\vec{v}\to-\vec{v}) \;=\; \frac{1}{\sqrt{2(p_0+m)}}(m+p_0-\vec{p}\cdot\vec{\sigma})\begin{pmatrix}1\\0\end{pmatrix}\frac{1}{\sqrt{2p_0}}e^{-ip_0(t+\vec{v}\cdot\vec{x})} \tag{6.103}$$

and (**ii**) reflect the spinor (6.103) in the mirror which changes only the space-time part of the wavefunction (described by e^{ipx}) but does not alter the spin part of the wavefunction (which transforms under Lorentz boosts according to eq. (6.63), see the discussion in previous Section):

$$[\xi^\alpha(\vec{v}\to-\vec{v})]^{\text{reflected}} \;=\; \frac{1}{\sqrt{2(p_0+m)}}(m+p_0-\vec{p}\cdot\vec{\sigma})\begin{pmatrix}1\\0\end{pmatrix}\frac{1}{\sqrt{2p_0}}e^{-ip_0(t+\vec{v}\cdot\vec{x})} \tag{6.104}$$

Thus, in this way, we obtain a different relativistic spinor

$$\eta_{\dot\alpha} = \frac{1}{\sqrt{2(p_0+m)}}(m+p_\mu\sigma^\mu\bar\sigma_0)_{\dot\alpha}^{\;\dot\beta}\kappa^{\dot\beta}\frac{1}{\sqrt{2p_0}}e^{-ipx} \tag{6.105}$$

which transforms according to (6.73) under Lorentz boosts. So, the parity reflection of our candidate for the wavefunction of spinor particle ξ^α transforms as our second candidate for the wavefunction of this particle $\eta_{\dot\alpha}$ (see eqs. (6.72), (6.73). The question is which of them shall we choose?

In the case of neutrino the asymmetry between the spinor and its reflection in the mirror was a reflection of the experimentally observed parity asymmetry of weak interactions (involving neutrinos). However, if we want to describe an electron, which is parity-even, this asymmetry is unconvenient. For example, let us try to describe the electron with momentum p and z -projection of the spin $\frac{1}{2}$ by the wavefunction (6.102). If we look in the mirror at the electron moving with velocity $-v$ we will see then the particle with wavefunction (6.105). On the other hand, physically this should be the same as if we look (without mirror) on an electron moving with velocity $+v$ which is described by the wavefunction (6.102). So, we have different formulas for the wavefunction one and the same physical electron which is no good. Therefore, we must find the description of the electron which respects parity. Actually, the most trivial way turns out to be the right one: let us describe the electron by a pair of the spinors (6.102) and (6.105) placed one above the other. We have then the wavefuncions

$$\psi^{(1)}(p) = \begin{pmatrix}\eta_{\dot\alpha}(p,\frac{1}{2})\\ \xi^\alpha(p,\frac{1}{2})\end{pmatrix} = \sqrt{\frac{1}{4p_0(p_0+m)}}\begin{pmatrix}(m+p_\mu\sigma^\mu\bar\sigma_0)\begin{pmatrix}1\\0\end{pmatrix}\\ (m+p_\mu\bar\sigma^\mu\sigma_0)\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}e^{-ipx} \tag{6.106}$$

for the spin up $(\frac{1}{2})$ and

$$\psi^{(2)}(p) = \begin{pmatrix} \eta_{\dot\alpha}(p,-\frac{1}{2}) \\ \xi^{\alpha}(p,-\frac{1}{2}) \end{pmatrix} = \sqrt{\frac{1}{4p_0(p_0+m)}} \begin{pmatrix} (m + p_\mu \sigma^\mu \bar\sigma_0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ (m + p_\mu \bar\sigma^\mu \sigma_0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} e^{-ipx} \tag{6.107}$$

for the electron with spin down $(-\frac{1}{2})$. One may object that the upper components of bispinor are defined in an unique way by the lower ones, so why is this doubling of writing? The answer is that this double-writing is convenient since under spatial reflection the upper and lower components of the bispinor (6.106) simply trade places so if we arrange our formalism to be symmetric under exchange of upper structures and lower structures this formalism will be explicitly parity-even. The two spinors $\xi^\alpha(p)$ and $\eta_{\dot\alpha}(p)$ satisfy a pair of conneted Weyl-type equations:

$$\begin{aligned} p_\mu(\sigma^\mu)_{\dot\alpha\alpha}\xi^\alpha(p) &= m\eta_{\dot\alpha}(p) \\ p_\mu(\bar\sigma^\mu)^{\alpha\dot\alpha}\eta_{\dot\alpha}(p) &= m\xi^\alpha(p) \end{aligned} \tag{6.108}$$

It is convenient to assemble them in one equation:

$$p_\mu \begin{pmatrix} 0 & \sigma^\mu_{\dot\alpha\beta} \\ (\bar\sigma^\mu)^{\alpha\dot\beta} & 0 \end{pmatrix} \begin{pmatrix} \eta_{\dot\beta}(p) \\ \xi^\beta(p) \end{pmatrix} = m \begin{pmatrix} \eta_{\dot\alpha}(p) \\ \xi^\alpha(p) \end{pmatrix} \tag{6.109}$$

where the elements of the 4×4 matrix in the l.h.s. of the eq. (6.109) are the 2×2 matrices. The four matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\dot\alpha\alpha} \\ (\bar\sigma^\mu)^{\alpha\dot\alpha} & 0 \end{pmatrix} \tag{6.110}$$

are called the Dirac matrices in the spinor representation and the equation

$$p_\mu(\gamma^\mu)_{\lambda\rho}u_\rho(p) = mu_\lambda(p) \tag{6.111}$$

is called the Dirac equation (in momentum space). Note that we've assembled two pairs of dotted and undotted indices in one Dirac subscript $u_\lambda = \left(\begin{smallmatrix} \eta_{\dot\alpha}(p) \\ \xi^\alpha(p) \end{smallmatrix}\right)$ (where λ=1,2,3,4). The solutions of this equation are called the Dirac bispinors (for the electron):

$$\begin{aligned} u^{(1)}(p) = \begin{pmatrix} \eta^{(1)}_{\dot\alpha} \\ \xi^\alpha_{(1)} \end{pmatrix} &= \frac{1}{\sqrt{2(p_0+m)}} \begin{pmatrix} (m + p_\mu \sigma^\mu \bar\sigma_0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ (m + p_\mu \bar\sigma^\mu \sigma_0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \sqrt{p\cdot\sigma\bar\sigma_0}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{p\cdot\bar\sigma\sigma_0}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \\ u^{(2)}(p) = \begin{pmatrix} \eta^{(1)}_{\dot\alpha} \\ \xi^\alpha_{(1)} \end{pmatrix} &= \frac{1}{\sqrt{2(p_0+m)}} \begin{pmatrix} (m + p_\mu \sigma^\mu \bar\sigma_0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ (m + p_\mu \bar\sigma^\mu \sigma_0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \sqrt{p\cdot\sigma\bar\sigma_0}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{p\cdot\bar\sigma\sigma_0}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \end{aligned} \tag{6.112}$$

It is convenient to define the so-called γ_5 matrix as follows

$$\gamma_5 \equiv \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \equiv \begin{pmatrix} -\delta^{\dot\alpha}_{\dot\beta} & 0 \\ 0 & \delta^\alpha_\beta \end{pmatrix} \tag{6.113}$$

so that $\frac{1-\gamma_5}{2}$ and $\frac{1+\gamma_5}{2}$ are projectors on the upper and lower sectors, e.g.

$$\frac{1-\gamma_5}{2}\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ 0 & 0 \end{pmatrix}, \qquad \frac{1+\gamma_5}{2}\gamma^\mu = \begin{pmatrix} 0 & 0 \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{6.114}$$

It is instructive to write down the relativistic invariant generalization of the operator of the spin of the electron in the rest frame. Suppose the spin of the electron in the rest frame is given by the vector

$$\vec{s} = \kappa^\dagger \vec{\sigma} \kappa \tag{6.115}$$

where κ is our spinor in the rest frame. Let us intoduce formally the four-vector s^μ which coincides with $(0, \vec{s})$ in the rest frame [57]. Note that $s^2 = -1$ and $s \cdot p = 0$.

With this notation the non-relativistic equation

$$\frac{1}{2}\vec{\sigma} \cdot \vec{s}\kappa^{(\vec{s})} = \frac{1}{2}\kappa^{(\vec{s})} \tag{6.116}$$

for the spinor $\kappa^{(\vec{s})}$ is generalized to

$$\frac{1}{2}\gamma_5\gamma_\mu s^\mu u(p,s) = \frac{1}{2}u(p,s) \tag{6.117}$$

where the spinor $u(p,s) \equiv u^{(\vec{s})}(p)$ is given by usual Lorentz transformation of the (bi)spinor $\begin{pmatrix} \kappa^{(\vec{s})} \\ \kappa^{(\vec{s})} \end{pmatrix}$

$$u(p,s) = \frac{1}{\sqrt{2(p_0+m)}} \begin{pmatrix} (m + p_0 - \vec{p} \cdot \vec{\sigma})\kappa^{(\vec{s})} \\ (m + p_0 + \vec{p} \cdot \vec{\bar{\sigma}})\kappa^{(\vec{s})} \end{pmatrix} \tag{6.118}$$

Indeed,

$$\gamma_5\gamma^\mu s_\mu = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} s_\mu = \begin{pmatrix} 0 & -\sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} s_\mu = \begin{pmatrix} 0 & -\sigma \cdot s \\ \bar{\sigma} \cdot s & 0 \end{pmatrix} \tag{6.119}$$

so in the rest frame the eq. (6.117) reduces to:

$$\frac{1}{2}\begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{s} \\ \vec{\bar{\sigma}} \cdot \vec{s} & 0 \end{pmatrix} \sqrt{m} \begin{pmatrix} \kappa^{(\vec{s})} \\ \kappa^{(\vec{s})} \end{pmatrix} = \lambda\sqrt{m} \begin{pmatrix} \kappa^{(\vec{s})} \\ \kappa^{(\vec{s})} \end{pmatrix} \Leftrightarrow \frac{1}{2}(\vec{\sigma} \cdot \vec{s})\kappa^{(\vec{s})} = \frac{1}{2}\kappa^{(\vec{s})} \tag{6.120}$$

which coincides with the eq. (6.116).

So, the two equations

$$\begin{aligned} p^\mu\gamma_\mu u(p,s) &= mu(p,s) \\ \gamma_5\gamma_\mu s^\mu u(p,s) &= u(p,s) \end{aligned} \tag{6.121}$$

fix the Dirac spinor unambigously.

Thus, the wavefunction (6.106) of the electron moving with momentum p and having spin $\lambda = \pm\frac{1}{2}$ in his rest frame in terms of Dirac spinors looks like

$$(\psi^e)^\lambda_{\vec{p}}(x) = \frac{u^\lambda(p)}{\sqrt{2p_0}}e^{-ipx} \tag{6.122}$$

[57] For example, $s = (0,1,0,0)$ for $\kappa = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $s = (0,0,1,0)$ for $\kappa = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$, and $s = (0,0,0,1)$ for $\kappa = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $s = (0,0,0,-1)$ for $\kappa = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as we already know.

so the positive-frequency part of the electron field has the form [58]

$$\psi_+(x) = \sum_{\lambda=\pm\frac{1}{2}} \int \frac{d^3p}{(2\pi)^3} \frac{u^\lambda(p)}{\sqrt{2p_0}} e^{-ipx} b(p,\lambda) \tag{6.123}$$

It is easy to see that $\psi_+(x)$ satisfies the *Dirac equation*

$$i\gamma^\mu \frac{d}{dx^\mu}\psi_+(x) = m\psi_+(x) \tag{6.124}$$

The negative-frequency part of the electron field should also satisfy this equation, so if we write this negative-frequency part in a form similar to (6.123):

$$\psi_-(x) = \sum_{\lambda=\pm\frac{1}{2}} \int \frac{d^3p}{(2\pi)^3} \frac{v^\lambda(p)}{\sqrt{2p_0}} d^*(p,\lambda) e^{ipx} \tag{6.125}$$

we see that the negative-frequency spinor v should satisfy the equation

$$p_\mu \gamma^\mu v(p) = -mv(p) \tag{6.126}$$

The solutions of this equation are:

$$\begin{aligned} v^{(1)}(p) &= \begin{pmatrix} \chi_{\dot\alpha}^{(1)} \\ \zeta^\alpha_{(1)} \end{pmatrix} = \frac{1}{\sqrt{2(p_0+m)}} \begin{pmatrix} (m + p_\mu\sigma^\mu\bar\sigma_0)\begin{pmatrix}1\\0\end{pmatrix} \\ (-m - p_\mu\bar\sigma^\mu\sigma_0)\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} \\ v^{(2)}(p) &= \begin{pmatrix} \chi_{\dot\alpha}^{(2)} \\ \zeta^\alpha_{(2)} \end{pmatrix} = \frac{1}{\sqrt{2(p_0+m)}} \begin{pmatrix} (-m - p_\mu\sigma^\mu\bar\sigma_0)\begin{pmatrix}0\\1\end{pmatrix} \\ (m + p_\mu\bar\sigma^\mu\sigma_0)\begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix} \end{aligned} \tag{6.127}$$

They are related to the positive-frequency solutions $u(p)$ by complex conjugation and multiplication by matrix $i\gamma_2$:

$$v^{(2)}(p) = i\gamma^2 \left(u^{(1)}(p)\right)^*, \quad v^{(1)}(p) = i\gamma^2 \left(u^{(2)}(p)\right)^* \tag{6.128}$$

(The role of the matrix $i\gamma_2$ is especially clear if we write down Dirac bispinor as a pair of Weyl spinors. Then the matrix $i\gamma_2$ simply converts the lower indices of Weyl spinors into upper ones and vice versa).

[58] here $u^{(\frac{1}{2})}(p) \equiv u^{(1)}(p)$, $u^{(-\frac{1}{2})}(p) \equiv u^{(2)}(p)$

Now we can write down the total expression for the classical electron field satisfying the Dirac equation:

$$\psi(x) = \sum_{\lambda=\pm\frac{1}{2}} \int \frac{d^3p}{(2\pi)^3} \left(\frac{u^\lambda(p)}{\sqrt{2p_0}} e^{-ipx} b(p,\lambda) + \frac{v^\lambda(p)}{\sqrt{2p_0}} e^{+ipx} d^*(p,\lambda) \right) \tag{6.129}$$

$$i\gamma^\mu \frac{d}{dx^\mu} \psi(x) = m\psi(x) \tag{6.130}$$

where $b(p,\lambda)$ and $d^*(p,\lambda)$ are some numerical functions. As in the case of π-mesons , the wavefunction of the antiparticle are related to the positive-frequency part of the complex conjugate field

$$\psi^\dagger(x) = \sum_{\lambda=\pm\frac{1}{2}} \int \frac{d^3p}{(2\pi)^3} \left(\frac{(v^\lambda)^\dagger(p)}{\sqrt{2p_0}} d(p,\lambda) e^{-ipx} + \frac{(u^\lambda)^\dagger(p,\lambda)}{\sqrt{2p_0}} b^*(p,\lambda) e^{ipx} \right) \tag{6.131}$$

It is convenient to define so-called Dirac conjugation

$$\bar\psi(x) = \psi^\dagger(x)\gamma_0 \tag{6.132}$$

then the expression for the Dirac conjugate field takes the form:

$$\bar\psi(x) = \sum_{\lambda=\pm\frac{1}{2}} \int \frac{d^3p}{(2\pi)^3} \left(\frac{\bar v^\lambda(p)}{\sqrt{2p_0}} d(p,\lambda) e^{-ipx} + \frac{\bar u^\lambda(p)}{\sqrt{2p_0}} b^*(p,\lambda) e^{ipx} \right) \tag{6.133}$$

Let us write down for completeness the Dirac equation for ψ and $\bar\psi$:

$$i\gamma^\mu \frac{d}{dx^\mu} \psi(x) = m\psi(x) \tag{6.134}$$

$$-i\frac{d}{dx^\mu} \bar\psi(x)\gamma^\mu = m\bar\psi(x) \tag{6.135}$$

So, the wavefunction of a freely moving positron with momentum p (and spin $\lambda = \pm\frac{1}{2}$ in the rest frame) is

$$\underline{\psi}^\lambda_{\bar p}(x) = \frac{\bar v^\lambda(p)}{\sqrt{2p_0}} e^{-ipx} \tag{6.136}$$

where the explicit form of the spinor $\bar v = (\chi^{\dot\alpha}, \zeta_\alpha)$ is:

$$\begin{aligned}
\bar v^{(\frac{1}{2})}(p) = \bar v^{(2)}(p) = \left(v^{(2)}(p)\right)^\dagger \gamma_0 = \tfrac{1}{\sqrt{2(p_0+m)}} \left((0,1)(m + \sigma_0\bar\sigma^\mu p_\mu); (0,1)(-m - \bar\sigma_0\sigma^\mu p_\mu)\right) \\
\bar v^{(-\frac{1}{2})}(p) = \bar v^{(1)}(p) = \left(v^{(1)}(p)\right)^\dagger \gamma_0 = \tfrac{1}{\sqrt{2(p_0+m)}} \left((1,0)(-m - \sigma_0\bar\sigma^\mu p_\mu); (1,0)(m + \bar\sigma_0\sigma^\mu p_\mu)\right)
\end{aligned} \tag{6.137}$$

The underlined notation for the positron wavefunction is a reminder that it is a row rather than a column (just as $\bar\psi^e$) [59].

[59] Note that ψ^p is a row unlike ψ^e which is a column. Of course, this is merely a convention - we could instead transpose ψ^p and it would be a column; but then the vertex for electron-positron annihilation into photon will be ugly.

We will need also the Dirac conjugates of the electron and positron wave function:

$$(\bar{\psi}^e)^{\lambda}_{\vec{p}}(x) \stackrel{\rm def}{\equiv} (\psi^p(x))^\dagger\gamma_0 = \frac{\bar{u}^\lambda(p)}{\sqrt{2p_0}}e^{ipx}, \qquad \breve{\psi}^p(x) \stackrel{\rm def}{\equiv} \gamma_0(\underline{\psi}^p(x))^\dagger = \frac{v^\lambda(p)}{\sqrt{2p_0}}e^{ipx} \tag{6.138}$$

Note that for the antiparticle we have $\bar{v}^{(\frac12)}(p) = \bar{v}^{(2)}(p)$, $\bar{v}^{(-\frac12)}(p) = \bar{v}^{(1)}(p)$ while for the particle the connection was simply $u^{(\frac12)}(p) = u^{(1)}(p)$, $u^{(-\frac12)}(p) = u^{(2)}(p)$. This is due to the fact that under 3–rotations the Hermitian conjugate spinors are transformed in the opposite way. To illustrate this, let us take an electron described by spinor $u^{(1)}(p)$ and a positron described by $\bar{v}^{(1)}(p)$ at rest, rotate them on angle ϕ around the OZ axis and compare the results. Since at rest the upper and lower parts of Dirac bispinor coincide, it is suffiient to compare the behavior of 2-component spinors $\begin{pmatrix}1\\0\end{pmatrix}$ and $(1,0)$ under this rotation. Since Hermitian conjugate spinors transform under rotations with the help of Hermitian conjugate matrices (see Eqs. (6.72) and (6.74)) we obtain

$$\begin{aligned} \xi^\alpha = \begin{pmatrix}1\\0\end{pmatrix} \;\; &\to \;\; \left(e^{-\frac{i}{2}\sigma_3\phi}\right)^\alpha{}_\beta \xi^\beta = e^{-i\phi/2}\begin{pmatrix}1\\0\end{pmatrix}, \\ (\xi^\alpha)^\dagger = (\xi^\dagger)^{\dot\alpha} = (1,0) \;\; &\to \;\; (\xi^\dagger)^{\dot\beta}\left(e^{\frac{i}{2}\sigma_3\phi}\right)_{\dot\beta}{}^{\dot\alpha} \;\; = \;\; e^{i\phi/2}(1,0) \end{aligned} \tag{6.139}$$

So, the Hermitian conjugate spinor $(1,0)$ describes (anti)particle with z-projection of the spin $= -\frac12$.

It is also clear from the transformation rules for Dirac spinors. The spinor $\psi = \begin{pmatrix}\eta^{(1)}_{\dot\alpha}\\ \xi^\alpha_{(1)}\end{pmatrix}$ is transformed under rotations as given by Eq. (6.72)

$$\psi = \begin{pmatrix}\eta^{(1)}_{\dot\alpha}\\ \xi^\alpha_{(1)}\end{pmatrix} = \begin{pmatrix}\left(e^{-i\frac12\vec\sigma\cdot\vec n\phi}\right)_{\dot\alpha}{}^{\dot\beta}\eta_{\dot\beta}\\ \left(e^{-i\frac12\vec\sigma\cdot\vec n\phi}\right)^\alpha{}_\beta \xi^\beta\end{pmatrix} \tag{6.140}$$

According to rules (6.72)-(6.74), the Dirac conjugate spinor

$$\bar\psi \;=\; \psi^\dagger\gamma_0 \;=\; \left((\eta^\dagger)_\beta, (\xi^\dagger)^{\dot\beta}\right)\begin{pmatrix}0 & \delta^\beta_\alpha\\ \delta^{\dot\alpha}_{\dot\beta} & 0\end{pmatrix} \;=\; ((\xi^\dagger)^{\dot\alpha}, (\eta^\dagger)_\alpha) \tag{6.141}$$

is transformed under rotations in a way opposite to Eq. (6.140).

$$((\xi^\dagger)^{\dot\alpha}, (\eta^\dagger)_\alpha) \;\;\to\;\; \left((\xi^\dagger)^{\dot\beta}\left(e^{i\frac12\vec\sigma\cdot\vec n\phi}\right)_{\dot\beta}{}^{\dot\alpha}, (\eta^\dagger)_\beta\left(e^{i\frac12\vec\sigma\cdot\vec n\phi}\right)^\beta{}_\alpha\right) \tag{6.142}$$

which gives Eq. (6.139) again. [60]

The explicit form of the Dirac conjugate spinor $\bar{u}^\lambda(p)$ is presented in the Appendix (see eq. (8.31). Using these explicit formulas for the Dirac spinors u and v it is easy to check

[60] Please note that in Eq. (6.141) we use $\gamma_0 = \begin{pmatrix}0 & \delta^\beta_\alpha\\ \delta^{\dot\alpha}_{\dot\beta} & 0\end{pmatrix}$ rather than $\gamma_0 = \begin{pmatrix}0 & \sigma^\mu_{\dot\alpha\alpha}\\ (\bar\sigma^\mu)^{\alpha\dot\alpha} & 0\end{pmatrix}$ - they are the same matrix $\gamma_0 = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$ but the labeling of indices is different.

the orthogonality conditions

$$\bar{u}^{\lambda}(p)u^{\lambda'}(p) = 2m\delta_{\lambda\lambda'} = -\bar{v}^{\lambda}(p)v^{\lambda'}(p) \tag{6.143}$$
$$\bar{u}^{\lambda}(p)\gamma^{\mu}u^{\lambda'}(p) = \bar{v}^{\lambda}(p)\gamma^{\mu}v^{\lambda'}(p) = 2p^{\mu}\delta_{\lambda\lambda'} \tag{6.144}$$
$$\bar{u}^{\lambda}(p)v^{\lambda'}(p) = 0 = \bar{v}^{\lambda}(p)u^{\lambda'}(p) \tag{6.145}$$

and the conditions of completeness (hereafter we use the common notation $\not{p} \stackrel{\text{def}}{\equiv} p_{\mu}\gamma^{\mu}$) [61]

$$\sum\nolimits_{\lambda=1,2}\left(u^{\lambda}_{\alpha}(p)\bar{u}^{\lambda}_{\beta}(p) - v^{\lambda}_{\alpha}(p)\bar{v}^{\lambda}_{\beta}(p)\right) = 2m\delta_{\alpha\beta} \tag{6.146}$$
$$\sum\nolimits_{\lambda=1,2} u^{\lambda}_{\alpha}(p)\bar{u}^{\lambda}_{\beta}(p) = (m+\not{p})_{\alpha\beta} \tag{6.147}$$
$$\sum\nolimits_{\lambda=1,2} v^{\lambda}_{\alpha}(p)\bar{v}^{\lambda}_{\beta}(p) = (\not{p}-m)_{\alpha\beta} \tag{6.148}$$

In a similar way one can prove that if we take a general spinor defined by 4-vector of spin s^{μ} according to eq. (6.121) we obtain that

$$\bar{u}(p,s)\gamma^{\mu}\gamma_5 u(p,s) = -\bar{v}(p,s)\gamma^{\mu}\gamma_5 v(p,s) = 2ms^{\mu} \tag{6.149}$$

Actually, the above equation can be used for the definition of the spinor (corresponding to 4-vector s^{μ}) instead of eq. (6.121), see Appendix D. Let us present also another useful formula:

$$\bar{u}(p,s)\gamma^{\mu}\gamma_5 u(p,-s) = \bar{v}(p,s)\gamma^{\mu}\gamma_5 v(p,-s) = 0 \tag{6.150}$$

It is worth noting that in many application the explicit form of spinors is not nessesary - the properties (6.143)-(6.150) are enough.

Sometimes it is convenient to specify not the relativistic invariant 4-vector $s_{\mu} = (s_0, \vec{s})$ but the helicity $\vec{s}\cdot\vec{p}$ of the electron in a certain frame [62].

The helicity operator is defined as

$$\hat{h} \stackrel{\text{def}}{\equiv} \frac{1}{2}\frac{\vec{p}}{|\vec{p}|}\cdot\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix} \tag{6.151}$$

The states with definite helicity $h = \pm\frac{1}{2}$ are the eigenstates of this operator. Indeed, the definition of the state with helicity h is that if we rotate it on the angle ϕ around the momentum $\vec{p}$ it is multiplied by the phase factor $e^{-ih\phi}$ where h is the helicity (eigenvalue of the helicity operator (6.151)). Since the operator of usual 3-dimensional rotation for the bispinor is simply

$$\begin{pmatrix} e^{-\frac{i}{2}\phi\frac{\vec{p}}{|\vec{p}|}\cdot\vec{\sigma}} & 0\\ 0 & e^{-\frac{i}{2}\phi\frac{\vec{p}}{|\vec{p}|}\cdot\vec{\sigma}}\end{pmatrix} = e^{-i\hat{h}\phi} \tag{6.152}$$

[61] When we are dealing with 4×4 matrices we define only indices of one sort - say, lower indices. This lower indices actually include both upper dotted and lower undotted indices of 2-spinors $u = \begin{pmatrix}\xi_{\dot{\alpha}}\\ \eta^{\alpha}\end{pmatrix}$ and $\bar{u} = (\xi^{\dot{\alpha}}, \eta_{\alpha})$. As usual, summation over the repeated indices is assumed.

[62] The helicity of the massive particle depends on the frame of reference, since one can always boost to a frame in which its momentum is in the opposite direction (but spin is unchanged). For a massless particle, which travels at the speed of light, one cannot perform such a boost so helicity is an inherent property of a massless particle.

we see that if the state is eigenfunction of the helicity operator the action of rotation operator on this state gives the phase factor $e^{-ih\phi}$. The explicit form of the spinors with definite helicity is

$$u^{[\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}} \begin{pmatrix} (m + p_\mu \sigma^\mu \bar{\sigma}_0)\omega^1 \\ (m + p_\mu \bar{\sigma}^\mu \sigma_0)\omega^1 \end{pmatrix}, \; u^{[-\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}} \begin{pmatrix} (m + p_\mu \sigma^\mu \bar{\sigma}_0)\omega^2 \\ (m + p_\mu \bar{\sigma}^\mu \sigma_0)\omega^2 \end{pmatrix} \tag{6.153}$$

where the two-component spinor ω has the form

$$\omega^{(1)} = \begin{pmatrix} e^{-i\alpha} \cos\left(\frac{\theta}{2}\right) \\ e^{i(\phi-\alpha)} \sin\left(\frac{\theta}{2}\right) \end{pmatrix}, \qquad \omega^{(2)} = \begin{pmatrix} -e^{-i\alpha} \sin\left(\frac{\theta}{2}\right) \\ e^{i(\phi-\alpha)} \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \tag{6.154}$$

where θ and ϕ are the polar and azimuthal angle of the momentum $\vec{p}$ and α is an arbitrary phase [63]. Note that the particular choice of spinors in Eq. (6.91) corresponds to $\omega^{(1)}$ at $\alpha = 0$ and in (6.92) to $\omega^{(2)}$ at $\alpha = \phi$.

Let us verify that the spinors (6.153) are the eigenstates of the helicity operator (6.151). First, it is easy to check that

$$\frac{1}{2}\frac{\vec{p}}{|\vec{p}|} \cdot \vec{\sigma}\omega^{(1)} = \frac{1}{2}\omega^{(1)}, \;\; \frac{1}{2}\frac{\vec{p}}{|\vec{p}|} \cdot \vec{\sigma}\omega^{(2)} = -\frac{1}{2}\omega^{(2)}, \tag{6.155}$$

Next, using Eqs. (6.155) we can reduce the spinors (6.153) to

$$u^{[\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}} \begin{pmatrix} (m + p_0 - |\vec{p}|)\omega^{(1)} \\ (m + p_0 + |\vec{p}|)\omega^{(1)} \end{pmatrix}, \; u^{[-\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}} \begin{pmatrix} (m + p_0 + |\vec{p}|)\omega^{(2)} \\ (m + p_0 - |\vec{p}|)\omega^{(2)} \end{pmatrix} \tag{6.156}$$

and therefore

$$\begin{aligned} \frac{1}{2}\frac{\vec{p}}{|\vec{p}|} \cdot \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \begin{pmatrix} (m + p_0 - |\vec{p}|)\omega^{(1)} \\ (m + p_0 + |\vec{p}|)\omega^{(1)} \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} (m + p_0 - |\vec{p}|)\omega^{(1)} \\ (m + p_0 + |\vec{p}|)\omega^{(1)} \end{pmatrix} \\ \frac{1}{2}\frac{\vec{p}}{|\vec{p}|} \cdot \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \begin{pmatrix} (m + p_0 + |\vec{p}|)\omega^{(2)} \\ (m + p_0 - |\vec{p}|)\omega^{(2)} \end{pmatrix} &= -\frac{1}{2} \begin{pmatrix} (m + p_0 + |\vec{p}|)\omega^{(2)} \\ (m + p_0 - |\vec{p}|)\omega^{(2)} \end{pmatrix}, \end{aligned} \tag{6.157}$$

Q.E.D.

The complete list for the spinors with definite helicity $u^h, \bar{u}^h, v^h, \bar{v}^h$ is presented in the Appendix (see eqs. (8.29)-(8.37)). The orthogonality and completeness conditions for the spinors with definite helicity have the same form (6.143)-(6.148) only one should replace $\lambda, \lambda' = \pm\frac{1}{2}$ by $h, h' = \pm\frac{1}{2}$.

The states with definite helicity are very convenient for the description of the high-energy processes since in the limit of large momenta they degenerate into the two-component spinors of a massless particle. Indeed, let us take the limit $|\vec{p}| \to \infty$ in the eq. (6.153). Then, for helicity $h = \pm\frac{1}{2}$ we obtain

$$u^{[\frac{1}{2}]}(p) \to \sqrt{2p_0} \begin{pmatrix} 0 \\ \omega^{(1)} \end{pmatrix}, \qquad u^{[-\frac{1}{2}]}(p) \to \sqrt{2p_0} \begin{pmatrix} \omega^{(2)} \\ 0 \end{pmatrix} \tag{6.158}$$

[63] In order to distinguish these spinors with helicity $\pm\frac{1}{2}$ from the spinors with z-component of the spin equal to $\pm\frac{1}{2}$ we put the helicity $\pm\frac{1}{2}$ in square brackets.

So, very fast electron with positive helicity behave almost as right-handed antineutrino and with negative helicity as left-handed neutrino (choose $\alpha = 0$ and compare to eq. (6.90), (6.91)). In a similar way, the fast-moving positron will have the wavefunction

$$\bar{v}^{[\frac{1}{2}]}(p) = -\sqrt{2p_0}\left(0, \quad \omega^{(2)\dagger}\right), \qquad \bar{v}^{[-\frac{1}{2}]}(p) = -\sqrt{2p_0}\left(\omega^{(1)\dagger}, \quad 0\right) \tag{6.159}$$

so the wavefunction of fast-moving positron with positive helicity resembes the (Dirac conjgate) wavefunction of right antineutrino. Similarly, negative–helicity positron looks like neutrino.

Homework assignment 6.

Problem 1. Write down the wavefunction of the electron moving in Y direction with speed v if we know that in the the rest frame the spin is pointing in X direction.

Problem 2. Write down the wavefunction of the positron moving in X direction with speed v if we know that in the the rest frame the spin is pointing in Y direction.

Part XXV

7 QED

7.1 Propagator of the electron

The Feynman Green function for the electron (and positron) can be constructed in a straightforward manner just as we have done in the case of (charged) π-mesons. The propagation amplitude is obtained using the general rule (3.36):

$$\tilde{K}^{e}_{\alpha\beta}(x-y) = \sum_{\lambda=\pm\frac{1}{2}} \int \frac{d^3p}{(2\pi)^3} (\psi^{\lambda}_{p})_{\alpha}(x)(\psi^{\lambda}_{p})^{\dagger}_{\beta}(y) \tag{7.1}$$

where $\lambda = \pm\frac{1}{2}$ are the components of the spin in the rest frame (or it could be helicities – the result after summation will be the same). I put $\psi^{\dagger}_{\beta}$ instead of ψ^{*}_{β} because we want our propagation function to be 4×4 matrix obeying the usual rules of matrix products (and then we must not only take complex conjugate of ψ^{λ}_{p} but transpose it as well). Furthermore, it is convenient to multiply this propagation amplitude by the unitary matrix γ_0 from the right (this corresponds to taking of Dirac conjugation of ψ), so finally the propagation amplitude of the electron takes the form:

$$K^{\mathrm{e}}_{\alpha\beta}(x-y) = \sum_{\lambda=\pm\frac{1}{2}} \int \frac{d^3p}{(2\pi)^3} (\psi^{\lambda}_{p})_{\alpha}(x)(\bar{\psi}^{\lambda}_{p})_{\beta}(y) = \sum_{\lambda=\pm\frac{1}{2}} \int \frac{d^3p}{(2\pi)^3 2p_0} (u^{\lambda}(p))_{\alpha}(\bar{u}^{\lambda}(p))_{\beta} e^{-ip(x-y)}\Big|_{p_0=E_p} \tag{7.2}$$

where $E_p = \sqrt{|\vec{p}|^2 + m^2}$ Using the completeness condition (6.146) we obtain:

$$K^{\mathrm{e}}_{\alpha\beta}(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_p} (m + \not{p})_{\alpha\beta} e^{-ip(x-y)}\Big|_{p_0=E_p} \tag{7.3}$$

Let us now derive the propagation amplitude of the positron. Repeating the same steps, we have (see eq. (6.138))

$$K^{\rm p}_{\alpha\beta}(x-y) =$$
$$\sum_{\lambda=\pm\frac{1}{2}} \int \frac{d^3p}{(2\pi)^3} \left((\underline{\psi}^{\rm p})^\lambda_p\right)_\alpha (x) \left((\breve{\psi}_{\rm p})^\lambda_p\right)_\beta (y) = \sum_{\lambda=\pm\frac{1}{2}} \int \frac{d^3p}{(2\pi)^3 2E_p} (\bar{v}^\lambda(p))_\alpha (v^\lambda(p))_\beta e^{-ip(x-y)}\Big|_{p_0=E_p}$$

As we shall see below, usually we need the transposed propagation amplitude

$$K^{\rm Tp}_{\alpha\beta}(x-y) = \tag{7.4}$$
$$\sum_{\lambda=\pm\frac{1}{2}} \int \frac{d^3p}{(2\pi)^3 2E_p} (v^\lambda(p))_\alpha (\bar{v}^\lambda(p))_\beta (y) e^{-ip(x-y)}\Big|_{p_0=\sqrt{|\vec{p}|^2+m^2}} = \int \frac{d^3p}{(2\pi)^3 2E_p} (\not{p}-m)_{\alpha\beta} e^{-ip(x-y)}\Big|_{p_0=E_p}$$

whe we have used the completeness condition (6.148)

Now we must restore Feynman functions from these propagation amplitudes. Let us recall how we have done it for the first time (for the scalar particle in external field) and repeat the same steps. Let us consider the scattering of the electron from the external potential shown in the diagram in Fig. 82 (for example, it may be the scattering of the electron from the Coulomb potential created by some heavy nucleus). Here the first contribution of

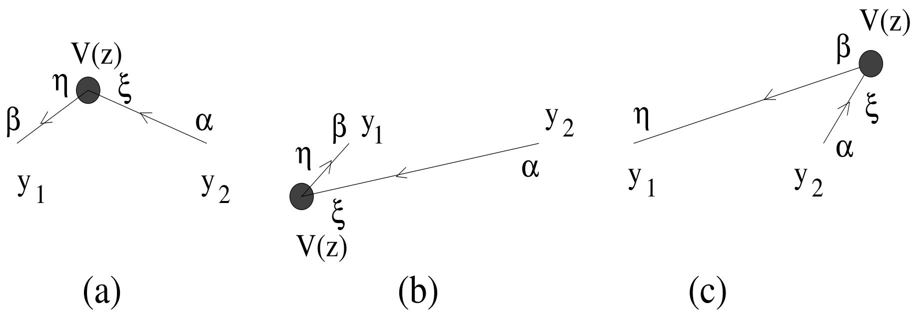


Figure 82. (Non-Feynman!) diagrams for the scattering of the electron by external potential. The direction of the flow of the charge is indicated by the arrow on the (solid) line.

diagram in Fig. 82a

$$\int dz K^{\rm e}_{\alpha\xi}(y_2-z)V_{\xi\eta}(z)K^{\rm e}_{\eta\beta}(z-y_1) \tag{7.5}$$

describes the situation when the electron was created in the point $\vec{r}_1$ at time $t = t_1 \equiv y_{10}$, propagated to the point $\vec{r}$ where at the the time $t \equiv z_0$ it interacted with the potential and finally was absorbed in the point $\vec{r}_2$ at the time $t_2 \equiv y_{20}$ [64] (we assume here that $y_{20} > y_{10}$). The second contribution (Fig. 82b)

$$\int dz K^{\rm e}_{\alpha\xi}(y_2-z)K^{\rm p}_{\beta\eta}(y_1-z)V_{\xi\eta}(z) = \int dz K^{\rm e}_{\alpha\xi}(y_2-z)V_{\xi\eta}(z)K^{\rm Tp}_{\eta\beta}(y_1-z) \tag{7.6}$$

[64] The interaction with the potential may depend on the spin of electron so $V(z)$ is in general 4×4 matrix with spinor indices.

corresponds to the situation when the external potential creates an electron-positron pair (not two electrons or positrons - it will contradict to the conservation of charge) at the moment of time $t < t_1, t_2$, they propagate and then one of them is annihilated in the point $\vec{r}_1$ at time t_1 and the second in $\vec{r}_2$ at time t_2. Similarly, the interpretation of the third term (see Fig. 82c)

$$\int dz V_{\xi\eta}(z) K^{\mathrm{e}}_{\eta\beta}(z-y_1) K^{\mathrm{p}}_{\xi\alpha}(z-y_2) = \int dz K^{\mathrm{Tp}}_{\alpha\xi}(z-y_2) V_{\xi\eta}(z) K^{\mathrm{e}}_{\eta\beta}(z-y_1) \tag{7.7}$$

is the following: the electron was created at $t_1, \vec{r}_1$ and the positron at $t_2, \vec{r}_2$ and after that they propagate to the point $t, \vec{r}$ where they had been absorbed by the potential. Now, the first term (7.5) is not relativistic invariant and, as in the scalar case, we would like to add the two additional contributions (7.6) and (7.6) in order to restore the relativistic invariance. However, in the case of electrons and positrons, we must subtract the two additional terms (7.6) and (7.6) rather than to add them. This property is related to the fact that inlike π-mesons, the electrons (and positrons) are Fermi particles obeying the Pauli principle, so you should not be surprised by the additional $(-)$ signs popping up now and then. I do not know the good explanation not based on second-quantization approach; one thing that I may say in my excuse is that, if we subtract those terms, we will get the relativistic invariant answer (see below) while if we add them, the result will not be relativistic invariant (and recall that in the scalar case we invented the addition of these terms in order to restore the relativistic invariance).

Thus, if we consider the difference of eqs. (7.5) and eqs. [(7.6)+(7.7)], it can be rewritten as follows:

$$\begin{aligned}\int dz \left(\Theta(y_{20}-z_0)K^{\mathrm{e}}(y_2-z) - \Theta(z_0-y_{20})K^{\mathrm{Tp}}(z-y_2)\right)_{\alpha\xi} V_{\xi\eta}(z) \\ \left(\Theta(z_0-y_{10})K^{\mathrm{e}}(z-y_1) - \Theta(y_{10}-z_0)K^{\mathrm{Tp}}(y_1-z)\right)_{\eta\beta}\end{aligned} \tag{7.8}$$

Now, the remarkable result is that the differences in the parentheses are relativistic invariant. Indeed, it is easy to show that

$$\begin{aligned}&\Theta(x_0-y_0)K^{\mathrm{e}}_{\alpha\beta}(x-y) - \Theta(y_0-x_0)K^{\mathrm{Tp}}_{\alpha\beta}(y-x) = \\ &\quad \Theta(x_0-y_0)\int \tfrac{d^3p}{(2\pi)^3 2E_p}(m+\not{p})_{\alpha\beta}e^{-ip(x-y)}\Big|_{p_0=E_p} - \Theta(y_0-x_0)\int \tfrac{d^3p}{(2\pi)^3 2E_p}(\not{p}-m)_{\alpha\beta}e^{-ip(y-x)}\Big|_{p_0=E_p} \\ &= \Theta(x_0-y_0)\int \tfrac{d^3p}{(2\pi)^3 2E_p}(m+\not{p})e^{-ip(x-y)}\Big|_{p_0=\sqrt{|\vec{p}|^2+m^2}} + \Theta(y_0-x_0)\int \tfrac{d^3p}{(2\pi)^3 2E_p}(m+\not{p})e^{-ip(x-y)}\Big|_{p_0=-E_p} \\ &\qquad = \int \tfrac{d^4p}{(2\pi)^4 i}(m+\not{p})_{\alpha\beta}\tfrac{1}{m^2-p^2-i\epsilon}e^{-ip(x-y)}\end{aligned} \tag{7.9}$$

(As usual, $E_p \equiv \sqrt{|\vec{p}|^2+m^2}$). The best way to prove the above equationis to take the final expression and perform the integration over p_0 by taking a residue. The eq. (7.9) is called the Feynman Green function of the electron (or positron – it describes both of them):

$$\begin{aligned}&G_0(x_2-x_1) = \\ &\Theta(x_{20}-x_{10})K^{\mathrm{e}}(x_2-x_1) - \Theta(x_{10}-x_{20})K^{\mathrm{Tp}}(x_1-x_2) = \int \tfrac{d^4p}{(2\pi)^4 i}(m+\not{p})\tfrac{1}{m^2-p^2-i\epsilon}e^{-ip(x_2-x_1)}\end{aligned} \tag{7.10}$$

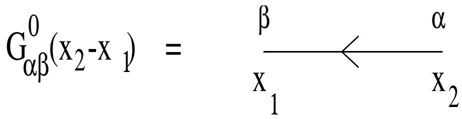


Figure 83. Feynman Green function.

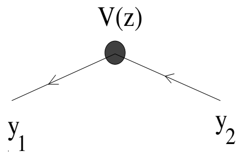


Figure 84. Feynman diagram for the scattering of electron from the external potential. Electron Green function is dentoted by the solid line with the arrow indicating the flow of charge.

Using this Feynman Green function, we can rewrite the sum (7.8) of the three (non-Feynman) diagrams in Fig. 82 in a usual form as the contribution of one Feynman diagram shown in Fig. 84

$$\int dz G_0(y_2 - z)V(z)G_0(z - y_1) \tag{7.11}$$

which is of course relativistic invariant. Consequently, same line represents both particle and antiparticle, depending on the sign of the time difference (and therefore in some textbooks positron is called "electron flying back in time", but I find such interpretation confusing). Actually, we could have derived the positron wavefunctions starting from this requirement that there should exist the common Green function describing both the propagation of electroms and positrons. (Recall that we have guessed the positron wavefunctions using the requirement that positron wavefunction correspond to the positive-frequency part of the conjugated Dirac field just as in the case of charged π-mesons. In this sense, the relativistic invariant formula (7.10) for the Feynman propagator is the justification of our guess).

Part XXVI

7.2 S-matrix and Green functions

As we noted above, the probability density for the electrons can be constructed in the following form:

$$\rho^e(t,R) = \psi^{e\dagger}(t,R)\psi^e(t,R) = \bar{\psi}^e(t,R)\gamma_0\psi^e(t,R) \tag{7.12}$$

where $\psi^e(x) \equiv \psi_+(x)$ (as usuall $x \equiv (t,r)$) is an arbitrary free-electron state — a superposition of the positive-frequency plane waves (see eq. (6.122)). Indeed, it is easy to demonstrate that

$$\frac{d}{dt}\rho(t,R) =$$
$$= \bar{\psi}^e(x)\gamma_0\tfrac{d}{dx_0}\psi^e(x) + \left(\tfrac{d}{dx_0}\bar{\psi}^e(x)\right)\gamma_0\psi^e(x) \;=\; \bar{\psi}^e(x)\left([-\gamma^i\tfrac{d}{dx^i} - im] + [-\gamma^i\tfrac{d}{dx^i} + im]\right)\psi^e(x)$$
$$= -\tfrac{d}{dx^i}\left(\bar{\psi}^e(x)\gamma^i\psi^e(x)\right) \tag{7.13}$$

where we have used the Dirac equations (6.134) and (6.135) for $\psi(x)$ and $\bar{\psi}(x)$. Therefore

$$\frac{d}{dt}\int d^3R\ \rho(t,R) \;=\; -\int d^3R\ \nabla_i\left(\bar{\psi}^e(x)\gamma_i\psi^e(x)\right) = 0 \tag{7.14}$$

after integration by parts. So, the quantity (7.12) meets our usual requirements for the denstity: it is positive-definite and conserved. [65] Similarly, the density of the positrons is

$$\rho_p(t,R) = \underline{\psi}^{p\dagger}(t,R)\underline{\psi}^p(t,R) \tag{7.15}$$

where

$$\underline{\psi}^p(x) = \sum_{\lambda=\pm\frac{1}{2}}\int\frac{d^3p}{(2\pi)^3}\frac{\bar{v}^\lambda(p)}{\sqrt{2p_0}}d(p,\lambda)e^{-ipx} \tag{7.16}$$

is a wavefunction for arbitrary free-positron wave packet. In order to put this to the matrix form similar to the r.h.s. of eq. (7.12) we will introduce the notation:

$$\breve{\psi}^p(x) \stackrel{\text{def}}{\equiv} \gamma_0(\underline{\psi}^p(x))^\dagger = \sum_{\lambda=\pm\frac{1}{2}}\int\frac{d^3p}{(2\pi)^3}\frac{v^\lambda(p)}{\sqrt{2p_0}}d(p,\lambda)e^{ipx} \tag{7.17}$$

then the positron density will reduce to the form similar to the electron one:

$$\rho_p(t,R) = \underline{\psi}^p(t,R)\gamma_0\breve{\psi}^p(t,R) \tag{7.18}$$

Because of the different form of the expression for the probability density (without time derivative, compare eqs. (5.3) and (7.12)) the formulas relating time evolution of the electron (and positron) states to the Green functions will be slightly different from the meson and photon cases. First, the orthogonality condition for the electron plane waves has the form (cf. (4.68)):

$$\int d^3r(\bar{\psi}^e)^\lambda_{\vec{p}}(t,\vec{r})\gamma_0(\psi^e)^{\lambda'}_{\vec{p}'}(t,\vec{r}) = (2\pi)^3\delta(\vec{p}-\vec{p}')\delta_{\lambda\lambda'} \tag{7.19}$$

and similarly for the positron plane waves:

$$\int d^3r(\underline{\psi}^p)^\lambda_{\vec{p}}(t,\vec{r})\gamma_0(\breve{\psi}^p)^{\lambda'}_{\vec{p}'}(t,\vec{r}) = (2\pi)^3\delta(\vec{p}-\vec{p}')\delta_{\lambda\lambda'} \tag{7.20}$$

It is easy to check these equations using the orthogonality property (6.144) for $\mu = 0$.

[65] The form of the probability density $\psi^{e\dagger}(t,R)\psi^e(t,R)$ is very similar to the non-relativistic density (3.14). This is due to the fact that Dirac equation (6.134) is of the first order in time derivative, just like the Schrödinger eqn (but unlike the second-order Klein-Gordon equation!)

Therefore, the time evolution for the one-particle state has the form:

$$
\begin{aligned}
(\psi^e)^\lambda_{\vec p}(t_2,\vec r_2) &= \int d^3r_1 K^{\rm e}(y_2-y_1)\gamma_0(\psi^e)^\lambda_{\vec p}(t_1,\vec r_1) = \int d^3r_1 G_0(y_2-y_1)\gamma_0(\psi^e)^\lambda_{\vec p}(t_1,\vec r_1) \\
(\underline{\psi}^p)^\lambda_{\vec p}(t_2,\vec R_2) &= \int d^3r_1(\underline{\psi}^p)^\lambda_{\vec p}(t_1,\vec r_1)\gamma_0 K^{\rm Tp}(y_2-y_1) = -\int d^3r_1(\underline{\psi}^p)^\lambda_{\vec p}(t_1,\vec r_1)\gamma_0 G_0(y_1-y_2)
\end{aligned}
\tag{7.21}
$$

where we used the general expressions for propagation functions (7.2) and (7.4) (for completeness, I will write them down them once more):

$$
\begin{aligned}
K^{\rm e}_{\alpha\beta}(x-y) &= \sum_{\lambda=\pm\frac12}\int\frac{d^3p}{(2\pi)^3}(\psi^\lambda_p)_\alpha(x)(\bar\psi^\lambda_p)_\beta(y) \\
K^{\rm Tp}_{\alpha\beta}(x-y) &= \sum_{\lambda=\pm\frac12}\int\frac{d^3p}{(2\pi)^3 2E_p}(\breve\psi^\lambda_p)_\alpha(y)(\underline{\psi}^\lambda_{-p})_\beta(x)
\end{aligned}
\tag{7.22}
$$

We see that the orthogonality conditions (7.19,7.20) and the one-particle time evolution (7.21) are similar to the corresponding formulas for the π-mesons (3.27) and (3.35) up to the replacement of the operator $i\overset{\leftrightarrow}{\frac{d}{dt}}$ by γ_0 (and taking care of spinor indices). Similarly, all the formulas for the relation between time evolution of the states and Green functions will have γ_0 instead of $i\overset{\leftrightarrow}{\frac{d}{dt}}$. For example, the two-electron component of the wavefunction of the state that was the two-electron state at $t=t_1$ has the form:

$$
\Psi^{ee}_{\xi_2,\xi_2'}(y_2,y_2') = \int d^3r_1 d^3r_1' G_{\xi_2,\xi_2';\xi_1,\xi_1'}(y_2,y_2';y_1,y_1')\left(\gamma_0\psi^e_{\vec p_1}(y_1)\right)_{\xi_1}\left(\gamma_0\psi^e_{\vec p_1'}(y_1')\right)_{\xi_1'}
\tag{7.23}
$$

where p_1,p_1' are initial momenta and $G_1(y_2,y_2';y_1,y_1')$ is a four-point Green function with four electron legs (here the dot-comma sign separates lines with incoming and outgoing charge). Another example is the electron-positron component of the $t=t_1$ electron-positron state

$$
\Psi_{ep}(y_2,y_2') = \int d^3r_1 d^3r_1'\left[(\underline{\psi}^p)_{\vec p_1'}(y_1')\gamma_0\right]_{\eta_1} G_{\xi_2,\eta_1;\xi_1,\eta_2}(y_2,y_1';y_1,y_2')\left[\gamma_0\psi^e_{\vec p_1}(y_1)\right]_{\xi_1}
\tag{7.24}
$$

Note that the four-point Green function with two electron and two positron legs is the same as the Green function with four electron legs up to relabeling of points and spinor indices, see Fig. 85 [66]

[66] The Green function in the coordinate space does not know whether the external tail belongs to particle or antiparticle, but it *does* know if the tail brings the charge or carries it out

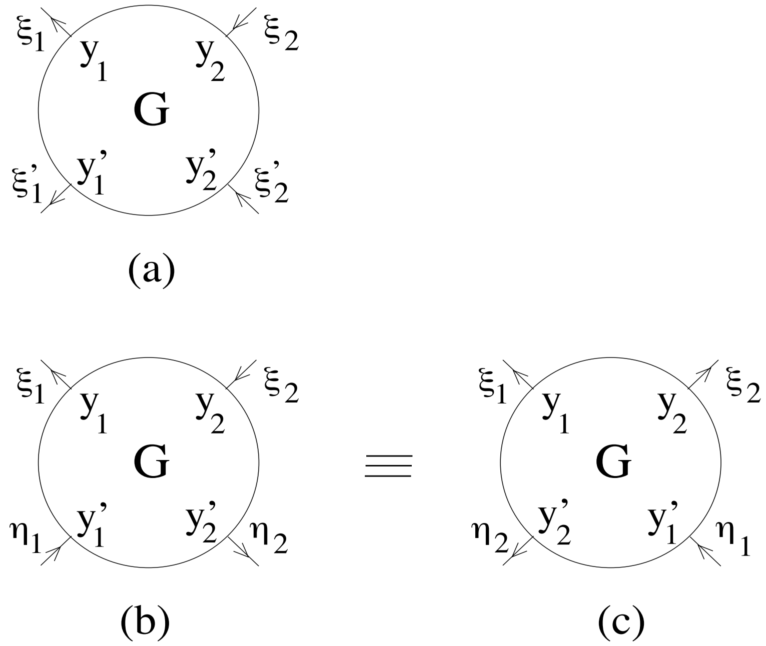


Figure 85. Four-fermion Green function for description of electron-electron scattering (a) and electron-positron scattering (b). Diagram (b) can be redrawn as diagram (c) which is the diagram (a) with relabeling of points and indices.

The projection onto the plane waves will also have extra γ_0's. For example, the element of the evolution matrix in the first (electron-electron case) is

$$U(t_2,t_1)^{\lambda_2\lambda_2';\lambda_1\lambda_1'}_{p_2,p_2';p_1,p_1'} = \int d^3r_1 d^3r_1' d^3r_2 d^3r_2' \tag{7.25}$$
$$\times \left[(\bar\psi^e)^{\lambda_2}_{\vec p_2}(y_2)\gamma_0\right]_{\xi_2}\left[(\bar\psi^e)^{\lambda_2'}_{\vec p_2'}(y_2')\gamma_0\right]_{\xi_2'} G(y_2,y_2';y_1,y_1')_{\xi_2,\xi_2',\xi_1,\xi_1'}\left[\gamma_0(\psi^e)^{\lambda_1}_{\vec p_1}(y_1)\right]_{\xi_1}\left[\gamma_0(\psi^e)^{\lambda_1'}_{\vec p_1'}(y_1')\right]_{\xi_1'}$$

while in our second example the U-matrix element is

$$U(t_2,t_1)^{\lambda_2\lambda_2';\lambda_1\lambda_1'}_{p_2,p_2';p_1,p_1'} = \int d^3r_1 d^3r_1' d^3r_2 d^3r_2' \tag{7.26}$$
$$\times \left[(\bar\psi^e)^{\lambda_2}_{\vec p_2}(y_2)\gamma_0\right]_{\xi_2}\left[\gamma_0(\breve\psi^p)^{\lambda_2'}_{\vec p_2'}(y_2')\right]_{\eta_2} G_{\xi_2,\eta_1,\xi_1,\eta_2}(y_2,y_1';y_1,y_2')\left[\gamma_0(\psi^e)^{\lambda_1}_{\vec p_1}(y_1)\right]_{\xi_1}\left[(\underline{\psi}^p)^{\lambda_1'}_{\vec p_1'}(y_1')\gamma_0\right]_{\eta_1}$$

Let us now take the limit $t_1 \to -\infty,\ t_2 \to \infty$ and obtain the relation between Green functions and matrix elements of the S-matrix. Consider for example the electron-positron scattering:

$$S^{\lambda_2\lambda_2';\lambda_1\lambda_1'}(p_2,p_2';p_1,p_1') = \lim_{t_1\to-\infty,\ t_2\to\infty} U(t_2,t_1)^{\lambda_2\lambda_2';\lambda_1\lambda_1'}_{p_2,p_2';p_1,p_1'} =$$
$$\lim_{t_1\to-\infty,\ t_2\to\infty}\int d^3r_1 d^3r_1' d^3r_2 d^3r_2'\left[((\bar\psi^e)^{\lambda_2}_{\vec p_2}(y_2)\gamma_0\right]_\alpha\left[\gamma_0(\breve\psi^p)^{\lambda_2'}_{\vec p_2'}(y_2')\right]_\xi$$
$$G(y_2,y_1';y_1,y_2')_{\alpha,\eta,\beta,\xi}\left[\gamma_0(\psi^e)^{\lambda_1}_{\vec p_1}(y_1)\right]_\beta\left[(\underline{\psi}^p)^{\lambda_1'}_{\vec p_1'}(y_1')\gamma_0\right]_\eta \tag{7.27}$$

Similarly to the case of scalar particles (4.94) we can represent the Green function as

$$
\begin{aligned}
G_{\alpha,\eta,\beta,\xi}(y_2,y_1';y_1,y_2') &= G^0_{\alpha\beta}(y_2-y_1)G^0_{\eta\xi}(y_1'-y_2') \qquad (7.28)\\
+\int dw_2dw_2'dw_1dw_1' G^0_{\alpha\alpha'}(y_2-w_2)&G^0_{\eta\eta'}(y_1'-w_1')G^{\mathrm{amp}}_{\alpha',\eta',\beta',\xi'}(w_2,w_1';w_1,w_2')G^0_{\beta'\beta}(w_1-y_1)G^0_{\xi'\xi}(w_2'-y_2')
\end{aligned}
$$

where the $G^{\mathrm{amp}}(w_2,w_2';w_1,w_1')$ is the Green function with amputated legs, see Fig. 86. The

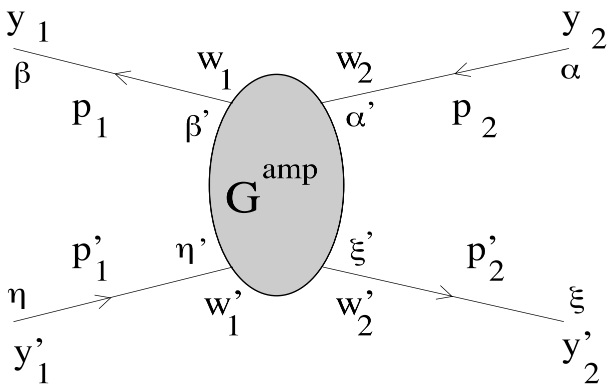


Figure 86. S-matrix from the Green function for the e^+e^- scattering

first term in r.h.s. of Eq. (7.28) describes the disconnected diagram without interaction. We will omit it in what follows.

As we have done many times before, since $t_1 \to -\infty$, $t_2 \to \infty$ we can replace each of the Green functions G_0 in eq. (7.28) by the corresponding propagation function K^e or K^p (see eq. (7.9)). Using formulas

$$
\begin{aligned}
\int d^3r_2\big[(\bar\psi^e)^{\lambda_2}_{\vec p_2}(y_2)\big]_\alpha K^e_{\alpha\alpha'}(y_2-w_2) &= \big[(\bar\psi^e)^{\lambda_2}_{\vec p_2}(w_2)\big]_{\alpha'}\\
\int d^3r_1'\big[(\underline{\psi}^p)^{\lambda_1'}_{\vec p_1'}(y_1')\gamma_0\big]_\eta K^{p\mathrm{T}}_{\eta\eta'}(w_1'-y_1') &= \big[(\underline{\psi}^p)^{\lambda_1'}_{\vec p_1'}(w_1')\big]_{\eta'}\\
\int d^3r_1 K^e_{\beta'\beta}(w_1-y_1)\big[\gamma_0(\psi^e)^{\lambda_1}_{\vec p_1}(y_1)\big]_\beta &= \big[(\psi^e)^{\lambda_1}_{\vec p_1}(w_1)\big]_{\beta'}\\
\int d^3r_2' K^{p\mathrm{T}}_{\xi'\xi}(y_2'-w_2')\big[(\gamma_0\breve\psi^p)^{\lambda_2'}_{\vec p_2'}(y_2')\big]_\xi &= \big[(\breve\psi^p)^{\lambda_2'}_{\vec p_2'}(w_2')\big]_{\xi'}
\end{aligned}
\qquad (7.29)
$$

(which can be easily verified using the relations (6.144)) we can reduce the expression (7.28) to the Fourier transform of the amputated Green function times some simple factors:

$$
\begin{aligned}
&S(p_2,p_2';p_1,p_1') =\\
&\frac{\bar u^{\lambda_2}_{\alpha'}(p_2)}{\sqrt{2E_2}}\frac{\bar v^{\lambda_1'}_{\eta'}(p_1')}{\sqrt{2E_1'}}G^{\mathrm{amp}}_{\alpha',\eta',\beta',\xi'}(p_1,p_1'\to p_2,p_2')\frac{u^{\lambda_1}_{\beta'}(p_1)}{\sqrt{2E_1}}\frac{v^{\lambda_2'}_{\eta'}(p_2')}{\sqrt{2E_2'}}\Big|_{p_2^2=(p_2')^2=p_1^2=(p_1')^2=m^2}
\end{aligned}
\qquad (7.30)
$$

So, we see that in comparison to the scattering of scalar particles (4.135), we must put additional factors

$$
\begin{array}{ll}
u^\lambda(p) & \text{for each incoming electron} \\
\bar{v}^\lambda(p) & \text{for each incoming positron} \\
\bar{u}^\lambda(p) & \text{for each outgoing electron} \\
v^\lambda(p) & \text{for each outgoing positron}
\end{array}
\tag{7.31}
$$

Part XXVII

7.3 Photon-electron interaction

Let us now find the form of elementary $ee\gamma$ vertex (see Fig. 88) in the same way as we have done for $\pi\pi\gamma$ vertex. Due to the Lorentz invariance this three-point Green function should

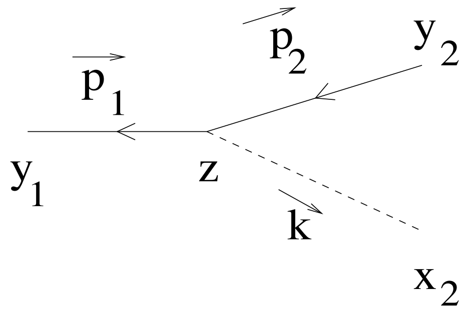


Figure 87. The $e \Rightarrow e\gamma$ transition.

have the form:

$$
G_\mu(x_2, y_2; y_1) = \int dz G_0(y_2 - z) D^0_{\mu\nu}(x_2 - z)\ \Gamma^\nu(z) G_0(z - y_1) \tag{7.32}
$$

Going to momentum space, we obtain:

$$
\mathcal{G}_\mu(p_1 - k, k; p_1) = \frac{m + \not{p}_1 - \not{k}}{m^2 - (p_1 - k)^2 - i\epsilon} \frac{1}{k^2 + i\epsilon} \Gamma_\mu \frac{m + \not{p}_1}{m^2 - p_1^2 - i\epsilon} \tag{7.33}
$$

Note that the vertex should have the Lorentz index μ reflecting the dependence of the $\pi \Rightarrow \pi\gamma$ amlitude on the polarization of the emitted photon. Because of the homogenuity of the space the vertex Γ_μ should not depend on the position z. On the other hand, the only vectors which does not depend on the position z is γ_μ, k_μ, and $p_\mu = (p_1 + p_2)_\mu$ (Due to the conservation of the momentum the third possibility $(p_1 - p_2)_\mu$ can be expressed via the first two). So, the most general form of this Green function is:

$$
\Gamma_\mu = a\gamma_\mu + bp_\mu + ck_\mu + d\gamma_\mu \not{p}_1 + d' \not{p}_2 \gamma_\mu \tag{7.34}
$$

where a,b,c,d, and d' are arbitrary numbers so far. (Two other possible terms $\sim \not{p}_1 \gamma_\mu$ and $\sim \gamma_\mu \not{p}_2$ reduce to the "d-terms" due to eq. (8.23)

$$
\not{p}_1 \gamma_\mu = p_1^\nu \gamma_\nu \gamma_\mu = p_1^\nu (2g_{\mu\nu} - \gamma_\mu \gamma_\nu) = 2p_\mu - \gamma_\mu \not{p}_1 \tag{7.35}
$$

and similarly for the second contribution $\sim \gamma_\mu \not{p}_2$).

Recall that the price to pay for the description of the photon by Feynman propagator (5.46) is Ward identity - after constructing the set of diagrams for QED we should prove that the longitudinal photons are not created. In QED the Ward identity is formulated as follows:

Suppose we have a general amputated Green function

$$\Pi_i \bar{u}(p_i)\Pi_j \bar{v}(p_j) G^{\rm amp}_{\mu_1,\ldots,\mu_m}(k_1,\ldots k_m,p_1,\ldots p_l)\Pi_m u(p_m)\Pi_n v(p_n)$$

with all the electron or positron momenta $p_1,\ldots p_l$ on the mass shell ($p_i^2=m^2$) and multiplied by the corresponding spinors u, $\bar{u}$,v, or $\bar{v}$ according to the rule (7.31). Then

$$k_l^{\mu_i}\Pi_i \bar{u}(p_i)\Pi_j \bar{v}(p_j) G^{\rm amp}_{\mu_1,\ldots,\mu_l,\ldots\mu_m}(k_1,\ldots k-l,\ldots k_m;p_1,\ldots p_l)\Pi_m u(p_m)\Pi_n v(p_n)=0 \tag{7.36}$$

Similarly to the case of π-meson electrodynamics, we will follow the following logic: first, we fix the elementary vertex Γ_μ by imposing the Ward identity (7.36) on Γ_μ, then construct the total set of Feynman diagrams, and finally prove the Ward identity (7.36) for the sum of all Feynman diagrams. So, the first step is to require that

$$k^\mu \bar{u}(p_2)\Gamma_\mu u(p_1)=0 \tag{7.37}$$

Before discussing this requirement, let us reduce the number of structures in the vertex (7.35). First, note that the term proporional to d can be rewritten as

$$d\gamma_\mu(\not{p}_1-m)+md\gamma_\mu \tag{7.38}$$

The second term in the r.h.s. leads to some redefinition of the constant a in the r.h.s. of eq. (7.34) while the first term vanishes due to the Dirac equation $(\not{p}_1-m)u(p_1)=0$. Similarly, one can get rid of the d' term so we obtain the general vertex in the form:

$$\Gamma_\mu=a\gamma_\mu+bp_\mu+ck_\mu \tag{7.39}$$

Now let us impose the condition (7.36):

$$k^\mu \bar{u}(p_2)(a\gamma_\mu+bp_\mu+ck_\mu)u(p_1)=0 \tag{7.40}$$

First, note that

$$\bar{u}(p-k)\not{k}\,u(p)=\bar{u}(p-k)[(m-\not{p}+\not{k})-(m-\not{p})]u(p)=0 \tag{7.41}$$

due to the Dirac equation. The second term also vanishes since

$$k^\mu p_\mu=k(2p_1-k)=p_1^2-p_2^2=0 \tag{7.42}$$

and both elecctrons are on the mass shell $p_1^2=p_2^2=m^2$. So, the condition (7.40) reduces to

$$k^2 c\bar{u}(p_2)u(p_1)=0 \tag{7.43}$$

whih means that $c = 0$ (recall that in order to avoid the trouble with the photon propagator we must require Ward identity for all photon momenta, not just for the real photons with $k^2 = 0$).

Therefore the most general form of the photoemission vertex (7.39) reduces to

$$\bar{u}(p_2)(a\gamma_\mu + bp_\mu)u(p_1) \tag{7.44}$$

It is instructive to rewrite this vertex in a different way. Using the so-called Gordon relation

$$\bar{u}(p-k)\gamma_\mu u(p) = \frac{(2p-k)_\mu}{2m}\bar{u}(p-k)u(p) - \frac{i}{2m}k^\nu \bar{u}(p-k)\sigma_{\mu\nu}u(p) \tag{7.45}$$

(where $\sigma_{\mu\nu} \stackrel{\text{def}}{\equiv} \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$) we can reduce eq. (7.44) to

$$\bar{u}(p_2)\left(A\gamma_\mu - \frac{i}{2m}B\sigma_{\mu\nu}k^\nu\right)u(p_1) \tag{7.46}$$

where $A = a + 2bm$ and $B = 2bm$. So, from general grounds the interaction of the electron with the photon can be described by two constants A and B which have a meaning of electric charge and anomalous magnetic moment of the electron, respectively (we will see it below). However, from the experiment we know that electron has no anomalous magnetic moment (only the composed particles such as nucleon do have it). So, the elementary vertex of QED is simply [67]

$$\Gamma_\mu(p_2, p_1) = A\gamma_\mu \tag{7.47}$$

Now we must show that the constant A is actually the electric charge of the electron. To this end we will calculate the amplitude of electron-proton scattering and compare it to Rutherford formula - just as we have done in π-meson electrodynamics. Let us neglect the effects due to the anomalous magnetic moment of the proton – they become important only at high electron energies comparable to the proton mass (940 MeV) and we want to compare to the non-relativistic Rutherford formula for small energies of the electron. Then proton for our purpose is just the massive positron with mass $M = 940MeV$. The first-order diagram for this scattering is shown in Fig. 88

The amputated reduced Green function for this diagram has the form:

$$\mathcal{G}^{\text{amp}}(p_2, p_2'; p_1, p_1') = \frac{A^2}{t + i\epsilon}\gamma_\mu \otimes \gamma^\mu \tag{7.48}$$

where $\gamma_\mu \otimes \gamma^\mu$ means $(\gamma_\mu)_{\alpha\beta}(\gamma^\mu)_{\xi\eta}$ – independent incdices. As we mentioned, the matrix elements of the S-matrix and T-matrix are obtained from the reduced Green functions just as in the case of scalar theory (see eq. (4.135) only with additional spinor factors (7.31). Therefore the matrix element of the T–matrix is:

$$T^{\lambda_2\lambda_2';\lambda_1\lambda_1'}(p_2, p_2'; p_1, p_1') = \frac{A^2}{t + i\epsilon}\bar{u}^{\lambda_2}(p_2)\gamma_\mu u^{\lambda_1}(p_1)\bar{V}^{\lambda_1'}(p_1')\gamma^\mu V^{\lambda_2'}(p_2') \tag{7.49}$$

[67] There is another form of the vertex compatible with gauge invariance, namely $A'\bar{u}(p_2)\gamma_\mu\gamma_5 u(p_1)$. However, this form is parity-odd (for example, at $p_2 = p_1$ we get the pseudovector of spin $\bar{u}(p_1)\gamma_\mu\gamma_5 u(p_1) = 2ms_\mu$) while the photon is parity-even so the coefficient A' will change sigh after reflection in the mirror $\Rightarrow$ $A' = 0$.

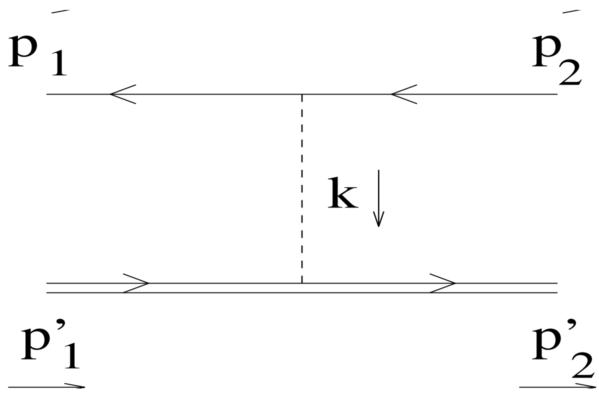


Figure 88. The elastic ep scattering. The proton is denoted by the double solid line with arrow pointing the direction of the charge flow.

where $V(p)$ means the proton spinor which is given by the usual formulas (8.29,8.30) and (8.32) with $m \to M$ replacement. The cross section of the electron-proton scattering in the lab frame is [68]

$$\left(\frac{d\sigma}{d\Omega}\right)^{\lambda_2\lambda_2';\lambda_1\lambda_1'} = \frac{1}{64\pi^2 M^2}|T^{\lambda_2\lambda_2';\lambda_1\lambda_1'}(p_2,p_2';p_1,p_1')|^2 \tag{7.51}$$

In order to compare to Rutherford formula let us calculate the cross section summed over final polarizations and averaged over the initial ones (both for electron and proton). This corresponds to the experiment in which polarizations of the particles are not registered. We have then

$$\frac{1}{4}\sum_{\lambda_1,\lambda_1'}\sum_{\lambda_2,\lambda_2'}|T^{\lambda_2\lambda_2';\lambda_1\lambda_1'}(p_2,p_2';p_1,p_1')|^2 = \frac{A^2}{4t^2}\sum_{\lambda_1,\lambda_1'}\sum_{\lambda_2,\lambda_2'}|\bar{u}^{\lambda_2}(p_2)\gamma_\mu u^{\lambda_1}(p_1)\bar{V}^{\lambda_1'}(p_1')\gamma^\mu V^{\lambda_2'}(p_2')|^2 \tag{7.52}$$

With the help of completeness relations (8.40) we obtain

$$\begin{aligned}
&\sum_{\lambda_1,\lambda_2}\bar{u}_\xi^{\lambda_2}(p_2)\gamma^\mu_{\xi\eta}u_\eta^{\lambda_1}(p_1)\bar{u}_\rho^{\lambda_1}(p_1)\gamma^\nu_{\rho\sigma}u_\sigma^{\lambda_2}(p_2) = \\
&\qquad (\not p_1+m)_{\eta\rho}\gamma^\nu_{\rho\sigma}(\not p_2+m)_{\sigma\xi}\gamma^\mu_{\xi\eta} = \mathrm{Tr}\left\{(\not p_1+\mathrm{m})\gamma^\nu(\not p_2+\mathrm{m})\gamma^\mu\right\}\\
&\sum_{\lambda_1',\lambda_2'}\bar{V}_\xi^{\lambda_1'}(p_1')\gamma^\mu_{\xi\eta}V^{\lambda_2'}(p_2')_\eta\bar{V}^{\lambda_2'}(p_2')_\rho\gamma^\nu_{\rho\sigma}V_\sigma^{\lambda_1'}(p_1') = \\
&\qquad (\not p_1'-M)_{\sigma\xi}\gamma^\mu_{\xi\eta}(\not p_2'-M)_{\eta\rho}\gamma^\nu_{\rho\sigma} = \mathrm{Tr}\left\{(\not p_1'-\mathrm{M})\gamma^\mu(\not p_2'-\mathrm{M})\gamma^\nu\right\}
\end{aligned} \tag{7.53}$$

[68] Since $m \ll M$ we may put $m=0$ and use the formula (8.16)

$$\left(\frac{d\sigma}{d\Omega}\right)^{\mathrm{Compton}}_{\mathrm{lab}} = \frac{|T|^2}{64\pi^2}\frac{1}{[M+p_1(1-\cos\theta)]^2} \tag{7.50}$$

Because we are interested in the non-relativistic limit, the momentum of the electron is also much less than M so this eqn reduces to (7.51).

so eq. (7.52) can be written as

$$\frac{A^2}{4t^2}\mathrm{Tr}\left\{(\not p_1+\mathrm{m})\gamma_\mu(\not p_2+\mathrm{m})\gamma_\nu\right\}\mathrm{Tr}\left\{(\not p_1'-\mathrm{M})\gamma^\mu(\not p_2'-\mathrm{M})\gamma_\nu\right\} \tag{7.54}$$

Using now the formula (8.25) for the trace of four γ-matrices we can reduce the product of traces in r.h.e of eq. (7.54) to

$$\begin{aligned}
&\mathrm{Tr}\left\{(\not p_1+\mathrm{m})\gamma_\mu(\not p_2+\mathrm{m})\gamma_\nu\right\}\mathrm{Tr}\left\{(\not p_1'-\mathrm{M})\gamma^\mu(\not p_2'-\mathrm{M})\gamma_\nu\right\} = \\
&= 16\left[p_1^\mu p_2^\nu+p_1^\nu p_2^\mu+(m^2-p_1\cdot p_2)g^{\mu\nu}\right]\left[p'_{1\mu}p'_{2\nu}+p'_{1\nu}p'_{2\mu}+(M^2-p'_1\cdot p'_2)g_{\mu\nu}\right] = \\
&= 32\left((p_2\cdot p'_2)(p_1\cdot p'_1)+(p_2\cdot p'_1)(p_1\cdot p'_2)-m^2(p'_1\cdot p'_2)-M^2(p_1\cdot p_2)+2M^2m^2\right) = \\
&= 16\left((s-M^2-m^2)^2+st+\frac{t^2}{2}\right)
\end{aligned} \tag{7.55}$$

Therefore, the eq.(7.52) takes the form

$$\begin{aligned}
&\frac{1}{4}\sum_{\lambda_1,\lambda_2}\sum_{\lambda'_1,\lambda'_2}|T^{\lambda_2\lambda'_2;\lambda_1\lambda'_1}(p_2,p'_2;p_1,p'_1)|^2 = \\
&\tfrac{4A^2}{t^2}\left[(s-M^2-m^2)^2+st+\tfrac{t^2}{2}\right]
\end{aligned} \tag{7.56}$$

so the final result for the (differential) cross section of the unpolarized *ep* scattering is:

$$\frac{d\sigma}{d\Omega}=\frac{A^4}{16\pi^2st^2}\left((s-M^2-m^2)^2+st+\frac{t^2}{2}\right) \tag{7.57}$$

In the low-energy limit when $p\stackrel{\mathrm{def}}{\equiv}|\vec p_i|\ll M$ (so $s-M^2-m^2\to 2p^2+2M\sqrt{p^2+m^2}$) we get the Mott formula for the scattering of the relativistic electrons from the Coulomb potential:

$$\frac{d\sigma}{d\Omega}=\frac{A^2}{4\pi^2st^2}M^2[p^2+m^2+\frac{t}{2}]=\frac{A^2}{64\pi^2v^2p^2\sin^4(\theta/2)}\left(1-v^2\sin^2(\theta/2)\right) \tag{7.58}$$

In the non-relativistic limit $v\ll 1$ the eq. (7.58) reduces to

$$\frac{d\sigma}{d\Omega}=\frac{A^2}{64\pi^2v^2p^2\sin^4(\theta/2)} \tag{7.59}$$

which is the Rutherford cross section (2.85). One can see now that the constant A is indeed the electric charge e.

Homework assignment 7.

Find the total cross section of $e^+e^-\to\mu^+\mu^-$ annihilation in the leading order in perturbation theory. The μ-meson has the same quantum numbers (charge, spin and parity) as electron but a different mass. Consider unpolarized electron and positron beams (average the result over the polarizations of initial electron and positron).

So, the elementary vertex of QED is $e\gamma_\mu$. Let us summarize Feynman rules for QED.

7.4 Final set of Feynman rules for QED

The Feynman rules for reduced Green function in the momentum space $\mathcal{G}(p_1, p_2, ...p_m, k_1, ...k_n)$ are:

I. Draw all different connected diagrams (without tadpoles). Electrons and positrons are depicted by straight line with the arrow indicating the flow of charge.

II. Draw momenta flow for each diagram taking into account momentum conservation in each vertex.

III. Each straight line with momentum p brings the factor $\mathcal{G}_0(-p) = \frac{m-\not{p}}{m^2-p^2-i\epsilon}$ if the chosen direction of flow of momentum p coincides with direction of the arrow on this line and the factor $\mathcal{G}_0(p) = \frac{m+\not{p}}{m^2-p^2-i\epsilon}$ if the directions of p and arrow are opposite.

IV. Each photon line brings the factor factor $\mathcal{D}_{\mu\nu}(p) \equiv \frac{g_{\mu\nu}}{k^2+i\epsilon}$

V.There is a factor $e\gamma_\mu$ for each vertex. Note that in our notations e is the *electron* charge so our vertex is $q_{\text{electron}}\gamma_\mu$. [69]

VI. There is an integration $\int \frac{d^4k}{(2\pi)^4 i}$ for each loop, and an extra factor (-1) for each fermion loop.

VI. There is no symmetry coefficients for QED, but there is a sign factor (instead). Assume sign $(+1)$ for the first graph that you have drawn and put (-1) between the diagrams which differ only by an interchange of two external identical fermion lines. This includes not only exchange of identical particles in the final state, but also interchange, for example, of initial particle and final antiparticle.

The transition matrix is the amputated Green function $\mathcal{G}(p_2, ...k_2^{n_2}; p_1, ..k_1^{n_1})$ on the mass shell times spinors and polarization vectors according to the rule (7.31) [70]. Suppose we consider scatering of m_1 electrons, n_1 positrons, and l_1 photons into of m_2 electrons, n_2 positrons, and l_2 photons. The corresponding term iof the transition matrix is

$$
\begin{aligned}
&T^{\lambda_2,...\lambda_2^{(l_2)};\lambda_1,...\lambda_1^{(l_1)}}_{h_2,...h_2^{(m_2)},H_2,...H_2^{(n_2)};h_1,...h_1^{(m_1)};H_1,...H_1^{(m_1)}}(p_2, ...p_2^{(m_2)}, q_2, ...q_2^{(n_2)}, k_2, ...k_2^{(n_2)}; p_1, ...p_1^{(m_1)}, q_1, ...q_2^{(n_1)}, k_1, ...k_1^{(n_1)}) \\
&= \ \Pi_{l_2} e^{\lambda_2^{(n_2)}}_{\mu_{(l_2)}}(k_2^{(l_2)})\Pi_{m_2}\bar{u}^{h_2}(p_2^{(n_2)})\Pi_{n_1}\bar{v}^{H_1}(q_1^{(n_1)})\Pi_{l_1} e^{\lambda_1^{(l_1)}}_{\mu^{l_1}}(k_1^{(n_1)})\Pi_{m_1}u^{h_1}(p_1^{(n_1)})\Pi_{n_2}v^{H_2}(q_2^{(n_2)}) \qquad (7.60) \\
&\times \ (\mathcal{G}^{\text{amp}})^{\mu_2,...\mu_2^{(l_2)};\mu_1,...\mu_1^{(l_1)}}(p_2, ...p_2^{(m_2)}, q_2, ...q_2^{(n_2)}, k_2, ...k_2^{(n_2)}; p_1, ...p_1^{(m_1)}, q_2, ...q_2^{(n_2)}, k_1, ...k_1^{(n_1)})\Big|_{p_i^2=m^2, k_i^2=0}
\end{aligned}
$$

where $p_1, ...p_1^{(m_1)}$ and $h_1, ...h_1^{(m_1)}$ are the momenta and the polarizations (or helicities) of initial electrons, $q_1, ...q_1^{(m_1)}$ and $H_1, ...H_1^{(m_1)}$ of initial positrons, and $k_1, ...k_1^{(n_1)}$ and $\lambda_1, ...\lambda_1^{(l_1)}$ of initial photons (and similarly for the final particles).

[69] In some textbooks (like *Peskin & Schroeder*) e denotes the positron charge, then the vertex is $-q_{\text{positron}}\gamma_\mu = -e\gamma_\mu$ in their notations.

[70] Amputation always mean removing the Green functions $G_0(p)$ corresponding to external legs. For the π -meson it was equivalent to multiplication by $(m^2 - p^2)$ (see eq. (4.135); for electrons and positrons it is equivalent to multiplication by $(m\mp \not{p})$ since $(m\mp \not{p})\frac{m\pm\not{p}}{m^2-p^2} = 1$.

7.5 Polarization effects in electron collisions

Let us consider the elastic scattering of electrons with positive helicities as an example. The two relevant diagrams are shown in Fig. 89

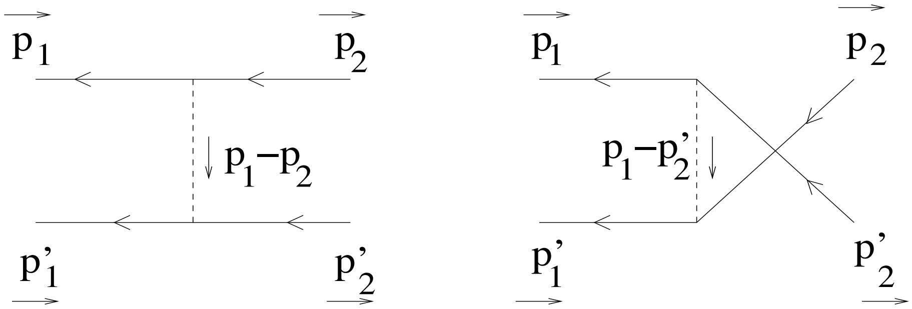


Figure 89. The elastic *ee* scattering .

The reduced Green function has the form:

$$\mathcal{G}(p_2, p_2'; p_1, p_1') = \frac{e^2}{t + i\epsilon}\gamma^\mu \otimes \gamma_\mu - \frac{e^2}{u + i\epsilon}\gamma^\mu \otimes \gamma_\mu \tag{7.61}$$

(note the (-) sign due to the exchange of identical electrons in the final state!). Therefore, the matrix element of the T-matrix is:

$$T^{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}(p_2, p_2'; p_1, p_1') =$$
$$\frac{e^2}{t+i\epsilon}\bar{u}^{(\frac{1}{2})}(p_2)\gamma^\mu u^{[\frac{1}{2}]}(p_1)\bar{u}^{[\frac{1}{2}]}(p_2')\gamma_\mu u^{[\frac{1}{2}]}(p_1') - \frac{e^2}{u+i\epsilon}\bar{u}^{[\frac{1}{2}]}(p_2')\gamma^\mu u^{[\frac{1}{2}]}(p_1)\bar{u}^{[\frac{1}{2}]}(p_2)\gamma_\mu u^{[\frac{1}{2}]}(p_1') \tag{7.62}$$

and so the cross section according to our general formula (8.9) takes the form:

$$\left(\frac{d\sigma}{d\Omega}\right)^{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}} = \frac{1}{64\pi^2 s}|T^{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}(p_2, p_2'; p_1, p_1')|^2 =$$
$$\frac{e^4}{64\pi^2 s}\Big(\frac{1}{t^2}\bar{u}^{[\frac{1}{2}]}(p_2)\gamma^\mu u^{[\frac{1}{2}]}(p_1)\bar{u}^{[\frac{1}{2}]}(p_1)\gamma^\nu u^{[\frac{1}{2}]}(p_2)\bar{u}^{[\frac{1}{2}]}(p_2')\gamma_\mu u^{[\frac{1}{2}]}(p_1')\bar{u}^{[\frac{1}{2}]}(p_1')\gamma_\nu u^{[\frac{1}{2}]}(p_2')$$
$$-\frac{2}{tu}\bar{u}^{[\frac{1}{2}]}(p_2)\gamma^\mu u^{[\frac{1}{2}]}(p_1)\bar{u}^{[\frac{1}{2}]}(p_1)\gamma^\nu u^{[\frac{1}{2}]}(p_2')\bar{u}^{[\frac{1}{2}]}(p_2')\gamma_\mu u^{[\frac{1}{2}]}(p_1')\bar{u}^{[\frac{1}{2}]}(p_1')\gamma_\nu u^{[\frac{1}{2}]}(p_2)$$
$$+\frac{1}{u^2}\bar{u}^{[\frac{1}{2}]}(p_2)\gamma^\mu u^{[\frac{1}{2}]}(p_1')\bar{u}^{[\frac{1}{2}]}(p_1')\gamma^\nu u^{[\frac{1}{2}]}(p_2)\bar{u}^{[\frac{1}{2}]}(p_2')\gamma_\mu u^{[\frac{1}{2}]}(p_1)\bar{u}^{[\frac{1}{2}]}(p_1)\gamma_\nu u^{[\frac{1}{2}]}(p_2')\Big) \tag{7.63}$$

Now we must calculate the product of spinors. It can be done by direct use of the formulas (8.32)-(8.35), but there is a trick that simplifies the calculations a lot.

Let us introduce the 4-vector of spin for the positive-helicity electron according to eq. (8.41):

$$\bar{u}^{[\frac{1}{2}]}(p)\gamma^\mu\gamma_5 u^{[\frac{1}{2}]}(p) = 2ms^\mu \tag{7.64}$$

then it is easy to see that

$$s^\mu(p, h = \frac{1}{2}) = \left(\frac{|\vec{p}|}{m}, \frac{\vec{p}p_0}{|\vec{p}|m}\right) \tag{7.65}$$

so the spatial component of 4-vector of spin is collinear to the direction of motion of the particle. (Similarly, for the helicity $-\frac{1}{2}$ the vector $s^\mu(p,h=-\frac{1}{2})=\frac{1}{2m}\bar{u}^{[-\frac{1}{2}]}(p)\gamma^\mu\gamma_5 u^{[-\frac{1}{2}]}(p)$ has the form:

$$s^\mu=\left(-\frac{|\vec{p}|}{m},-\frac{\vec{p}p_0}{|\vec{p}|m}\right) \tag{7.66}$$

since the spin is opposite to the direction of motion). Let introduce the so-called projection operators:

$$\Lambda_+=\frac{1+\gamma_5\not{s}}{2},\quad \Lambda_-=\frac{1-\gamma_5\not{s}}{2} \tag{7.67}$$

They have the properties

$$\Lambda_+^2=\Lambda_+,\quad \Lambda_-^2=\Lambda_-,\ \Lambda_+\Lambda_-=0,\quad \Lambda_++\Lambda_-=1 \tag{7.68}$$

It is easy to verify that

$$\Lambda_+u^{[\frac{1}{2}]}(p)=u^{[\frac{1}{2}]}(p),\quad \Lambda_+u^{[-\frac{1}{2}]}(p)=0 \tag{7.69}$$

Then we can replace

$$u^{[\frac{1}{2}]}(p_1)\bar{u}^{[\frac{1}{2}]}(p_1)=\Lambda_+(s_1)\sum_h u^{[\frac{1}{2}]}(p_1)\bar{u}^{[\frac{1}{2}]}(p_1)=\frac{1+\gamma_5\not{s}_1}{2}(m+\not{p}_1) \tag{7.70}$$

where s_1 is the 4-vector of spin (7.65) for the first electron. After such replacement we can use the completeness relation (8.40) for each product of spinors in eq. (7.63).

We have then:

$$\begin{aligned}
&\left(\frac{d\sigma}{d\Omega}\right)^{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}=\frac{1}{64\pi^2 s}|T^{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}(p_2,p_2';p_1,p_1')|^2=\\
&=\frac{e^4}{1024\pi^2 s}\Big(\frac{1}{t^2}\mathrm{Tr}\{\gamma^\mu(1+\gamma_5\not{s}_1)(\mathrm{m}+\not{p}_1)\gamma^\nu(1+\gamma_5\not{s}_2)(\mathrm{m}+\not{p}_2)\}\mathrm{Tr}\{\gamma^\mu(1+\gamma_5\not{s}_1')(\mathrm{m}+\not{p}_1')\gamma^\nu(1+\gamma_5\not{s}_2')(\mathrm{m}+\not{p}_2')\}\\
&-\frac{2}{tu}\mathrm{Tr}\{\gamma^\mu(1+\gamma_5\not{s}_1)(\mathrm{m}+\not{p}_1)\gamma^\nu(1+\gamma_5\not{s}_2')(\mathrm{m}+\not{p}_2')\gamma_\mu(1+\gamma_5\not{s}_1')(\mathrm{m}+\not{p}_1')\gamma_\nu(1+\gamma_5\not{s}_2)(\mathrm{m}+\not{p}_2)\}\\
&+\frac{1}{u^2}\mathrm{Tr}\{\gamma^\mu(1+\gamma_5\not{s}_1')(\mathrm{m}+\not{p}_1')\gamma^\nu(1+\gamma_5\not{s}_2)(\mathrm{m}+\not{p}_2)\}\mathrm{Tr}\{\gamma_\mu(1+\gamma_5\not{s}_1)(\mathrm{m}+\not{p}_1)\gamma_\nu(1+\gamma_5\not{s}_2')(\mathrm{m}+\not{p}_2')\}\Big)
\end{aligned} \tag{7.71}$$

where

$$\begin{aligned}
s_1&=\left(\tfrac{p}{m},\tfrac{\vec{p}_1E_1}{pm}\right), & s_1'&=\left(\tfrac{p}{m},-\tfrac{\vec{p}_1E_1}{pm}\right)\\
s_2&=\left(\tfrac{p}{m},\tfrac{\vec{p}_2E_1}{m}\right), & s_2'&=\left(\tfrac{p}{m},-\tfrac{\vec{p}_2E_1}{pm}\right)
\end{aligned} \tag{7.72}$$

and $p\equiv|\vec{p_1}|=|\vec{p_1'}|=|\vec{p_2}|=|\vec{p_2'}|$.

The calculation of traces of γ- matrices in eq. (7.71) is straightforward but tedious. (Quite often for the calculation of traces of 8 or more γ matrices people use computers). We will consider consider the simpler case of high-energy scattering when

$$|\vec{p_1}|\gg m \tag{7.73}$$

Then the vector of spin is approximately collinear to momentum

$$|\vec{p}_1| \gg m \Rightarrow s_\mu(p,h=\frac{1}{2}) = \left(\frac{p}{m}, \frac{\vec{p}_1 E_1}{pm}\right) \simeq \frac{p_\mu}{m}, \quad s_\mu(p,h=-\frac{1}{2}) \simeq -\frac{p_\mu}{m} \tag{7.74}$$

Therefore the projection operators (8.41) reduce to:

$$\begin{aligned} u^{[\frac{1}{2}]}(p)\bar{u}^{[\frac{1}{2}]}(p) = & \ \tfrac{1}{2}\left(1+\gamma_5 \ \not{s}(p,\tfrac{1}{2})\right)(m+\not{p}) \ \rightarrow \frac{1}{2}(1+\gamma_5)(m+\not{p}) \\ u^{[-\frac{1}{2}]}(p)\bar{u}^{[-\frac{1}{2}]}(p) = & \ \tfrac{1}{2}\left(1+\gamma_5 \ \not{s}(p,-\tfrac{1}{2})\right)(m+\not{p}) \rightarrow \frac{1}{2}(1-\gamma_5)(m+\not{p}) \end{aligned} \tag{7.75}$$

For completeness, let us present similar formulas for the projection operators for high-energy positrons (see Eq. (8.42) from Appendix C):

$$\begin{aligned} v^{[\frac{1}{2}]}(p)\bar{v}^{[\frac{1}{2}]}(p) = & \ \tfrac{1}{2}\left(1+\gamma_5 \ \not{s}(p,\tfrac{1}{2})\right)(\not{p}-m) \ \rightarrow \frac{1}{2}(1-\gamma_5)(\not{p}-m) \\ v^{[-\frac{1}{2}]}(p)\bar{v}^{[-\frac{1}{2}]}(p) = & \ \tfrac{1}{2}\left(1+\gamma_5 \ \not{s}(p,-\tfrac{1}{2})\right)(\not{p}-m) \rightarrow \frac{1}{2}(1+\gamma_5)(\not{p}-m) \end{aligned} \tag{7.76}$$

The projector $\Lambda_R \stackrel{\text{def}}{\equiv} \frac{1+\gamma_5}{2}$ kills the lower components of the Dirac bispinor and the projector $\Lambda_L \stackrel{\text{def}}{\equiv} \frac{1-\gamma_5}{2}$ kills the upper ones (The name "right" and "left" for these projectors is due to the fact that the bispinor $\Lambda_R\xi$ has only upper components and transforms like the 2-spinor for the right antineutrino while the bispinor $\Lambda_L\xi$ has only lower components which transform like the 2-spinor for left neutrino.)

The projectors Λ_R, Λ_L satisfy the usual properties

$$\Lambda_R^2 = \Lambda_R, \quad \Lambda_L^2 = \Lambda_L, \quad \Lambda_R\Lambda_L = 0, \quad \text{and} \quad \Lambda_R + \Lambda_L = 1 \tag{7.77}$$

to calculate the traces in the high-energy limit (7.73) it is convenient to return to eq. (7.71). Using the property

$$\Lambda_R\gamma_\mu = \gamma_\mu\Lambda_L \tag{7.78}$$

and the set of properties (7.77) it is easy to show that all the terms in eq. (7.71) proportional to mass m vanish (as one should expect at large energies $\gg m$) so the answer for the cross section reduces to:

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)^{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}_{\text{high-energy}} = & \\ & \frac{e^4}{64\pi^2 s}\Big(\tfrac{1}{t^2}\text{Tr}\{\gamma^\mu\tfrac{1+\gamma_5}{2} \ \not{p}_1\gamma^\nu \ \not{p}_2\}\text{Tr}\{\gamma^\mu\tfrac{1+\gamma_5}{2} \ \not{p}'_1\gamma^\nu \ \not{p}'_2\} \\ -\tfrac{2}{tu}\text{Tr}\{\gamma^\mu\tfrac{1+\gamma_5}{2} \ \not{p}_1\gamma^\nu \ \not{p}'_2\gamma_\mu \ \not{p}'_1\gamma_\nu \ \not{p}_2)\} & + \tfrac{1}{\text{u}^2}\text{Tr}\{\gamma^\mu\tfrac{1+\gamma_5}{2} \ \not{p}'_1\gamma^\nu \ \not{p}_2\}\text{Tr}\{\gamma_\mu\tfrac{1+\gamma_5}{2} \ \not{p}_1\gamma_\nu \ \not{p}'_2\}\Big) \end{aligned} \tag{7.79}$$

Now it is easy to calculate the relevant traces. First, the longest trace is the easiest since

$$\begin{aligned} \text{Tr}\{\gamma^\mu\tfrac{1+\gamma_5}{2} \ \not{p}_1\gamma^\nu \ \not{p}'_2\gamma_\mu \ \not{p}'_1\gamma_\nu \ \not{p}_2)\} = -2\text{Tr}\{\tfrac{1+\gamma_5}{2} \ \not{p}_1 \ \not{p}'_1\gamma_\mu \ \not{p}'_2 \ \not{p}_2\gamma^\mu\} = \\ -8(p_2\cdot p'_2)\text{Tr}\{\tfrac{1+\gamma_5}{2} \ \not{p}_1 \ \not{p}'_1\} = -16(\text{p}_1\cdot\text{p}'_1)(\text{p}_2\cdot\text{p}'_2) \end{aligned} \tag{7.80}$$

where we have used the properties (8.24). Other traces are also not difficult. We obtain:

$$\mathrm{Tr}\{\gamma_\mu \frac{1+\gamma_5}{2} \not{p}_1 \gamma_\nu \not{p}_2\} = 2\left(\mathrm{p}_{1\mu}\mathrm{p}_{2\nu} + \mathrm{p}_{2\mu}\mathrm{p}_{1\nu} - \mathrm{g}_{\mu\nu}\mathrm{p}_1\cdot\mathrm{p}_2 + \mathrm{i}\epsilon_{\mu\nu\alpha\beta}\mathrm{p}_1^\alpha \mathrm{p}_2^\beta\right) \tag{7.81}$$

so

$$\begin{aligned}
\mathrm{Tr}\{\gamma^\mu \frac{1+\gamma_5}{2} \not{p}_1 \gamma^\nu \not{p}_2\}\mathrm{Tr}\{\gamma^\mu \frac{1+\gamma_5}{2} \not{p}'_1 \gamma^\nu \not{p}'_2\} &= 16(p_1\cdot p'_1)(p_2\cdot p'_2)\\
\mathrm{Tr}\{\gamma^\mu \frac{1+\gamma_5}{2} \not{p}_1 \gamma^\nu \not{p}'_2\}\mathrm{Tr}\{\gamma^\mu \frac{1+\gamma_5}{2} \not{p}'_1 \gamma^\nu \not{p}_2\} &= 16(p_1\cdot p'_1)(p_2\cdot p'_2)
\end{aligned} \tag{7.82}$$

(when multiplying the traces, we used eq. (8.26)). At large energies $(p_1\cdot p'_1) = (p_2\cdot p'_2) \simeq \frac{s}{2}$ since $s \gg m^2$. Thus, the cross section reduces to

$$\left(\frac{d\sigma}{d\Omega}\right)^{\frac12\frac12,\frac12\frac12}_{\mathrm{high-energy}} = \frac{e^4}{16\pi^2}s\left(\frac{1}{t}+\frac{1}{u}\right)^2 \tag{7.83}$$

In a similar way, one can show that

$$\begin{aligned}
\left(\tfrac{d\sigma}{d\Omega}\right)^{-\frac12,-\frac12;-\frac12,-\frac12}_{\mathrm{high-energy}} &= \tfrac{e^4}{16\pi^2 s}\left(\tfrac{s}{t}+\tfrac{s}{u}\right)^2\\
\left(\tfrac{d\sigma}{d\Omega}\right)^{\frac12,-\frac12;\frac12,-\frac12}_{\mathrm{high-energy}} = \left(\tfrac{d\sigma}{d\Omega}\right)^{-\frac12,\frac12;-\frac12,\frac12}_{\mathrm{high-energy}} &= \tfrac{e^4}{16\pi^2 s}\left(\tfrac{u}{t}\right)^2\\
\left(\tfrac{d\sigma}{d\Omega}\right)^{\frac12,-\frac12;-\frac12,\frac12}_{\mathrm{high-energy}} = \left(\tfrac{d\sigma}{d\Omega}\right)^{-\frac12,\frac12;\frac12,-\frac12}_{\mathrm{high-energy}} &= \tfrac{e^4}{16\pi^2 s}\left(\tfrac{t}{u}\right)^2
\end{aligned} \tag{7.84}$$

while all other amplitudes are zero

$$\left(\tfrac{d\sigma}{d\Omega}\right)^{\frac12,\frac12;\frac12,-\frac12}_{\mathrm{high-energy}} = \left(\tfrac{d\sigma}{d\Omega}\right)^{\frac12,\frac12;-\frac12,-\frac12}_{\mathrm{high-energy}} = \left(\tfrac{d\sigma}{d\Omega}\right)^{\frac12,-\frac12;-\frac12,-\frac12}_{\mathrm{high-energy}} = \left(\tfrac{d\sigma}{d\Omega}\right)^{\frac12,-\frac12;\frac12,\frac12}_{\mathrm{high-energy}} = 0 \tag{7.85}$$

At very high energies, when $s \gg t$ (and $t \gg m^2$) all t he (nonzero) amplitudes are equal:

$$\left(\tfrac{d\sigma}{d\Omega}\right)^{\frac12,\frac12;\frac12,\frac12}_{\mathrm{Regge}} = \left(\tfrac{d\sigma}{d\Omega}\right)^{-\frac12,-\frac12;-\frac12,-\frac12}_{\mathrm{Regge}} = \left(\tfrac{d\sigma}{d\Omega}\right)^{\frac12,\frac12;-\frac12,-\frac12}_{\mathrm{Regge}} = \tfrac{e^4 s}{16\pi^2 t^2} \tag{7.86}$$

while the $\frac12,-\frac12;-\frac12,\frac12$ cross section joins the majority of vanishing contributions. This simple structure of the answer is the reflection of the conservation of helicity in Regge limit (Regge limit means $s \gg t, m^2$).

Part XXVIII

7.6 Compton scattering

The cross section of Compton scattering from the electron in the lab frame is given by the formula (8.16):

$$\left(\frac{d\sigma}{d\Omega}\right)^{\mathrm{Compton}}_{\mathrm{lab}} = \frac{|T|^2}{64\pi^2}\frac{1}{[m+k_1(1-\cos\theta)]^2} \tag{7.87}$$

For the unpolarized scattering, we must sum over the final polarizations (of electrons and photons) and average over the initial ones:

$$\left(\frac{d\sigma}{d\Omega}\right)^{\text{unpolarized}} = \frac{1}{4}\sum_{\lambda_1,h_1}\sum_{\lambda_2,h_2}\frac{|T^{\lambda_1,h_1\,\lambda_2,h_2}(p_2,k_2;p_1,k_1)|^2}{64\pi^2}\frac{1}{[m_b+k_1(1-\cos\theta)]^2} \tag{7.88}$$

where λ are the photon and electron polarizations. So, we must calculate

$$|T^2|^{\text{unpolarized}} \stackrel{\text{def}}{\equiv} \frac{1}{4}\sum_{\lambda_1,h_1}\sum_{\lambda_2,h_2}|T^{\lambda_1,h_1,\lambda_2,h_2}(p_2,k_2;p_1,k_1)|^2 \tag{7.89}$$

As usual, we start by drawing the relevant diagrams - see Fig. 90

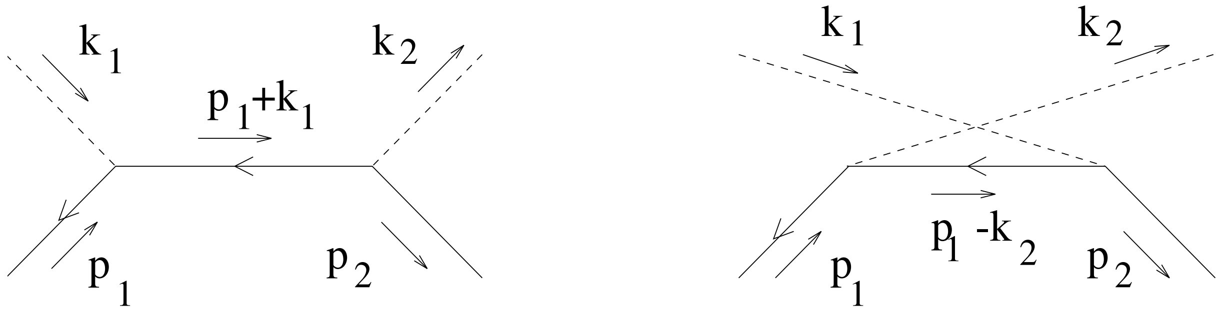


Figure 90. Lowest-order diagrams for the Compton scattering from the electron.

Second step is to find the reduced amputated Green function which according to our set of Feynman rules has the form:

$$\mathcal{G}_{\mu\nu}(p_2,k_2;p_1,k_1) = \frac{\gamma_\nu(m+\not{p}_1+\not{k}_1)\gamma_\mu}{m^2-s-i\epsilon} + \frac{\gamma_\mu(m+\not{p}_1-\not{k}_2)\gamma_\nu}{m^2-u-i\epsilon} \tag{7.90}$$

so the matrix element of the T-matrix takes the form (see the rule (7.60)):

$$T^{\lambda_2,h_2;\lambda_1,h_1}(p_2,k_2;p_1,k_1) = \\ \frac{\bar{u}^{h_2}(p_2)\not{e}^{\lambda_2}(k_2)(m+\not{p}_1+\not{k}_1)\not{e}^{\lambda_1}(k_1)u^{h_1}(p_1)}{m^2-s-i\epsilon} + \frac{\bar{u}^{h_2}(p_2)\not{e}^{\lambda_1}(k_1)(m+\not{p}_1-\not{k}_2)\not{e}^{\lambda_2}(k_2)u^{h_1}(p_1)}{m^2-u-i\epsilon} \tag{7.91}$$

where h_1, h_2 and $\lambda_1\lambda_2$ are the initial and final polarizations for the electron and photon, respectively.

Therefore

$$|T^2|^{\text{unpolarized}} = \frac{1}{4}\sum_{\lambda_1,h_1}\sum_{\lambda_2,h_2} \\ \left(\frac{\bar{u}^{h_2}(p_2)\not{e}^{\lambda_2}(k_2)(m+\not{p}_1+\not{k}_1)\not{e}^{\lambda_1}(k_1)u^{h_1}(p_1)}{m^2-s-i\epsilon} + \frac{\bar{u}^{h_2}(p_2)\not{e}^{\lambda_1}(k_1)(m+\not{p}_1-\not{k}_2)\not{e}^{\lambda_2}(k_2)u^{h_1}(p_1)}{m^2-u-i\epsilon}\right) \\ \left(\frac{\bar{u}^{h_1}(p_1)\not{e}^{\lambda_1}(k_1)(m+\not{p}_1+\not{k}_1)\not{e}^{\lambda_2}(k_2)u^{h_2}(p_2)}{m^2-s+i\epsilon} + \frac{\bar{u}^{h_1}(p_1)\not{e}^{\lambda_2}(k_2)(m+\not{p}_1-\not{k}_2)\not{e}^{\lambda_1}(k_1)u^{h_2}(p_2)}{m^2-u+i\epsilon}\right) \tag{7.92}$$

If we choose the polarizations as shown in Fig. 91 we have the property

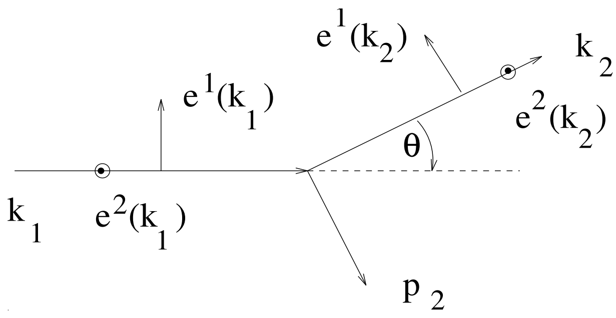


Figure 91. Kinematics for the Compton scattering.

$$e^{(2)}(k_i)\cdot p_j = 0, \quad e^{(1)}(k_i)\cdot p_1 = 0 \tag{7.93}$$

(and of course $e^{(i)}(k_1)\cdot k_1 = e^{(i)}(k_2)\cdot k_2 = 0$). Then the expression (7.92) reduces to

$$\begin{aligned} |T^2|^{\text{unpolarized}} = \frac{1}{4}\sum_{\lambda_1,h_1}\sum_{\lambda_2,h_2} \\ \left(\frac{\bar{u}^{h_2}(p_2)\not{e}^{\lambda_2}\ \not{k}_1\ \not{e}^{\lambda_1}u^{h_1}(p_1)}{m^2-s-i\epsilon} - \frac{\bar{u}^{h_2}(p_2)\not{e}^{\lambda_1}\ \not{k}_2\ \not{e}^{\lambda_2}u^{h_1}(p_1)}{m^2-u-i\epsilon}\right) \\ \left(\frac{\bar{u}^{h_1}(p_1)\not{e}^{\lambda_1}\ \not{k}_1\ \not{e}^{\lambda_2}u^{h_2}(p_2)}{m^2-s+i\epsilon} - \frac{\bar{u}^{h_1}(p_1)\not{e}^{\lambda_2}\ \not{k}_2\ \not{e}^{\lambda_1}u^{h_2}(p_2)}{m^2-u+i\epsilon}\right) \end{aligned} \tag{7.94}$$

where we used the property

$$(m+\not{p}_1)\ \not{e}(k_i)u(p) = \not{e}(k_i)(m-\not{p}_1)u(p) = 0, \quad \bar{u}(p)\ \not{e}(k_i)(m+\not{p}_1) = \bar{u}(p)(m-\not{p}_1)\ \not{e}(k_i) = 0 \tag{7.95}$$

After summation over electron polarizations according to (8.40) one obtains (note that $s-m^2 = 2p_1\cdot k_1 = 2p_2\cdot k_2$ and $u-m^2 = 2p_1\cdot k_2 = 2p_2\cdot k_1$, and $t = -2k_1\cdot k_2 = 2m^2 - 2p_1\cdot p_2$):

$$\begin{aligned} &|T^2|^{\text{unpolarized}} = \\ &\tfrac{1}{2}\textstyle\sum_{\lambda_1,\lambda_2}\Big(\tfrac{1}{(2p_1\cdot k_1)^2}\mathrm{Tr}\{\not{e}^{\lambda_2}\ \not{k}_1\ \not{e}^{\lambda_1}(\mathrm{m}+\not{p}_1)\ \not{e}^{\lambda_1}\ \not{k}_1\ \not{e}^{\lambda_2}(\mathrm{m}+\not{p}_2)\} \\ &+\tfrac{2}{(2p_1\cdot k_1)(2p_1\cdot k_2)}\mathrm{Tr}\{\not{e}^{\lambda_2}\ \not{k}_1\ \not{e}^{\lambda_1}(\mathrm{m}+\not{p}_1)\ \not{e}^{\lambda_2}\ \not{k}_2\ \not{e}^{\lambda_1}(\mathrm{m}+\not{p}_2)\} + \\ &\qquad \tfrac{1}{(2p_1\cdot k_2)^2}\mathrm{Tr}\{\not{e}^{\lambda_1}\ \not{k}_2\ \not{e}^{\lambda_2}(\mathrm{m}+\not{p}_1)\ \not{e}^{\lambda_2}\ \not{k}_2\ \not{e}^{\lambda_1}(\mathrm{m}+\not{p}_2)\}\Big) \end{aligned} \tag{7.96}$$

The relevant traces are:

$$\frac{1}{4}\mathrm{Tr}\{\not{e}^{\lambda_2}\ \not{k}_1\ \not{e}^{\lambda_1}(\mathrm{m}+\not{p}_1)\ \not{e}^{\lambda_1}\ \not{k}_1\ \not{e}^{\lambda_2}(\mathrm{m}+\not{p}_2)\} = 2k_1\cdot p_1\left(k_2\cdot p_1 + 2(k_1\cdot e^{\lambda_2})^2\right) \tag{7.97}$$

$$\begin{aligned} \frac{1}{4}\mathrm{Tr}\{\not{e}^{\lambda_1}\ \not{k}_2\ \not{e}^{\lambda_2}(\mathrm{m}+\not{p}_1)\ \not{e}^{\lambda_2}\ \not{k}_2\ \not{e}^{\lambda_1}(\mathrm{m}+\not{p}_2)\} &= 2k_2\cdot p_1\left(k_1\cdot p_1 - 2(k_2\cdot e^{\lambda_1})^2\right) \\ \frac{1}{4}\mathrm{Tr}\{\not{e}^{\lambda_2}\ \not{k}_1\ \not{e}^{\lambda_1}(\mathrm{m}+\not{p}_1)\ \not{e}^{\lambda_2}\ \not{k}_2\ \not{e}^{\lambda_1}(\mathrm{m}+\not{p}_2)\} &= 2(k_1\cdot p_1)(k_2\cdot p_1)\left(2(e^{\lambda_1}\cdot e^{\lambda_2})^2 - 1\right) \\ &\quad - 2(k_1\cdot e^{\lambda_2})^2 k_2\cdot p_1 + 2(k_2\cdot e^{\lambda_1})^2 k_1\cdot p_1 \end{aligned}$$

(The calculation of the third trace is somewhat lengthly).

Putting all the traces together in eq. (7.96) we find the square of the T-matrix for unpolarized electron in the form:

$$\left(|T|^2\right)^{\text{unp electron}} = 4(e^{\lambda_1}\cdot e^{\lambda_2})^2 - 2 + \frac{k_2\cdot p_1}{k_1\cdot p_1} + \frac{k_1\cdot p_1}{k_2\cdot p_1} \tag{7.98}$$

(we will perform the summation over photon polarization later). The corresponding expression for the cross section is known as the Klein-Nishina formula for the Compton scattering from unpolarized electron

$$\left(\frac{d\sigma}{d\Omega}\right)^{\text{unp electron}} = \frac{e^4}{64\pi^2 m^2}\left(\frac{|\vec{k}_2|}{|\vec{k}_1|}\right)^2\left[\frac{|\vec{k}_2|}{|\vec{k}_1|} + \frac{|\vec{k}_1|}{|\vec{k}_2|} + 4\left(e^{\lambda_1}(k_1)\cdot e^{\lambda_2}(k_2)\right)^2 - 2\right] \tag{7.99}$$

where $\frac{|\vec{k}_1|}{|\vec{k}_2|} = 1 + \frac{|\vec{k}_1|}{m}(1-\cos\theta)$, see eq. (8.15) from Appendix B.

In the non-relativistic limit ($|\vec{k}_1| \ll m$) this reduces to the classical Thomson scattering:

$$\left(\frac{d\sigma}{d\Omega}\right)^{\text{unp electron}}_{\vec{k}_1\to 0} = \frac{e^4}{16\pi^2 m^2}\left(e^{\lambda_1}(k_1)\cdot e^{\lambda_2}(k_2)\right)^2 \tag{7.100}$$

where

$$\frac{e^2}{4\pi mc^2} = 2.8\times 10^{-13} cm \tag{7.101}$$

is the classical electron radius (in usual units).

The unpolarized cross section is obtained by final summation over λ_2 and averaging over λ_1, so we get

$$\left(\frac{d\sigma}{d\Omega}\right)^{\text{unpolarized}} = \frac{1}{2}\sum_{\lambda_1,\lambda_2}\left(\frac{d\sigma}{d\Omega}\right)^{\text{unp electron}} = \frac{e^4}{32\pi^2 m^2}\left(\frac{|\vec{k}_2|}{|\vec{k}_1|}\right)^2\left[\frac{|\vec{k}_2|}{|\vec{k}_1|} + \frac{|\vec{k}_1|}{|\vec{k}_2|} - \sin^2\theta\right] \tag{7.102}$$

8 Appendix

8.1 Cross section for general $2 \Rightarrow 2$ scattering in the c.m. frame.

Let us calculate the differential cross section in the c.m. frame for a general $2 \Rightarrow 2$ transition. Suppose we have two particles A and B with masses M_A and M_B in the initial state and two particles a and b with masses m_a and m_b in the final state. We will calculate the differential cross section

$$\frac{d\sigma}{d\Omega} \tag{8.1}$$

for the particle a (which has mass m_a) to fly into the spherical angle $d\Omega$. (Because of the momentum conservation, the particle b will fly in the opposite direction). We can start from the eq. (4.112) - everything up to this equation can be repeated without alteration for our general case.

$$d\sigma = \frac{1}{I}\frac{d^3p_a}{(2\pi)^3 4E_aE_b}2\pi\delta(E_a + E_b - E_A - E_B)|T(p_a, p_b; p_A, p_B)|^2 \tag{8.2}$$

where the invariant flux has the form (cf. eq. (4.109):

$$I = 4E_AE_B\left(\frac{|\vec{p}_A|}{E_A} + \frac{|\vec{p}_B|}{E_B}\right) = 4\sqrt{(p_A \cdot p_B)^2 - M_A^2M_B^2} = 2\sqrt{(s - M_A - M_B)^2 - 4M_A^2M_B^2} \tag{8.3}$$

As usual, in the c.m. frame we have $|\vec{p}_A| = |\vec{p}_B|$ and due to the conservation of momentum we have that $|\vec{p}_a| = |\vec{p}_b| \stackrel{\text{def}}{\equiv} p_2$ also. Therefore the expression (8.2) can be written down as:

$$d\sigma = \frac{1}{I}\frac{p_2^2dp_2d\Omega}{(2\pi)^3 4E_aE_b}2\pi\delta(\sqrt{m_a^2 + p_2^2} + \sqrt{m_b^2 + p_2^2} - \sqrt{s})|T(p_a, p_b; p_A, p_B)|^2 \tag{8.4}$$

(recall that the Mandelstam variable s in the c.m. frame is simply $(E_A + E_B)^2$ Now, we must perform the integration over p_2 using the δ-function

$$\delta(\sqrt{m_a^2 + p_2^2} + \sqrt{m_b^2 + p_2^2} - \sqrt{s}) = \left(\frac{p_\star}{\sqrt{p_\star^2 + m_a^2}} + \frac{p_\star}{\sqrt{p_\star^2 + m_b^2}}\right)^{-1}\delta(p_2 - p_\star) \tag{8.5}$$

where $p_\star$ is the solution of the equation:

$$\sqrt{m_a^2 + p_\star^2} + \sqrt{m_b^2 + p_\star^2} = \sqrt{s} \tag{8.6}$$

It is easy to to see that the explicit form of $p_\star$ is

$$p_\star = \frac{1}{2\sqrt{s}}\sqrt{(s - m_a^2 - m_b^2)^2 - 4m_a^2m_b^2} \tag{8.7}$$

With formulas (8.5) and (8.7) at hand, the integration over p_2 in eq. (8.4) becomes trivial so we get

$$\frac{d\sigma}{d\Omega} = \frac{p_\star^2}{16\pi^2E_aE_bI}\left(\frac{p_\star}{E_a} + \frac{p_\star}{E_b}\right)^{-1}|T|^2 = \frac{p_\star}{16\pi^2(E_a + E_b)I}|T|^2 \tag{8.8}$$

Recalling the formulas (8.7), (8.3) and the fact that $E_a + E_b = \sqrt{s}$ we obtain the final expression for the cross section in the c.m. frame:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{\sqrt{(s-m_a^2-m_b^2)^2-4m_a^2m_b^2}}{\sqrt{(s-M_A^2-M_B^2)^2-4M_A^2M_B^2}}|T|^2 = \frac{1}{64\pi^2 s}\frac{|\vec{p}_a|}{|\vec{p}_A|}|T|^2 \tag{8.9}$$

Writing the r.h.s. we used the formula (8.7) for $p_\star = |\vec{p}_a|$ $(= |\vec{p}_b|)$ and similar formula for $|\vec{p}_A|$ $(= |\vec{p}_B|)$ which expresses it through the invariant s and masses M_A and M_B:

$$|\vec{p}_A| = \frac{1}{2\sqrt{s}}\sqrt{(s-M_A^2-M_B^2)^2-4M_A^2M_B^2} \tag{8.10}$$

This finishes the kinematical part of the job. The rest is the calculation of the corresponding element of the transition matrix $T(p_a, p_b; p_A, p_B)$.

8.2 Cross section of elastic scattering in the lab frame

Suppose we have a particle with mass m_a scattered from the particle with mass m_b which is originally originally at rest. We measure the cross section for this m_a particle to fly in the spherical angle $d\Omega$ (see Fig. 92) Let us derive the formula for the expression of this

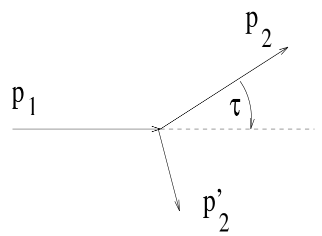


Figure 92. The elastic scattering in lab frame

lab-framee cross section in terms of T-matrix (similar to c.m. frame formula (8.9)). Again, we can repeat the calculation from nthe previous section up to eq. (8.4), instead of which we will get [71]

$$d\sigma = \frac{1}{I}\frac{p_2^2 dp_2 d\Omega}{(2\pi)^3 4E2E_2'}2\pi\delta\left(\sqrt{m_a^2+p_2^2}+\sqrt{m_b^2+p_1^2+p_2^2-2p_1p_2\cos\theta}-E_1-m_b\right)|T(p_1,p_1';p_2,p_2')|^2 \tag{8.11}$$

(see the kinematics in Fig. (92). Performing the integration over p_2 with the help of δ-function (see formula(1.4), we obtain (recall that $I = 4p_1m_b$, see eq. (8.3)):

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm lab} = \frac{p_*^2}{64\pi^2 p_1 m_b}\frac{1}{p_*(E_1+m_a)-p_1E_2\cos\theta}|T|^2 \tag{8.12}$$

where $E_2 = \sqrt{p_*^2+m_b^2}$ and p_* is the solution of the equation

$$\sqrt{m_a^2+p_*^2}+\sqrt{m_b^2+p_1^2+p_*^2-2p_1p_*\cos\theta}-E_1-m_b = 0 \tag{8.13}$$

[71] In the agrument of T-matrix p_1, p_2 (and p_1', p_2') mean the 4-vectors while everywhere else $p_2 \equiv |\vec{p}_2|$, $p_1 \equiv |\vec{p}_1|$

The explicit form of p_* is:

$$p_* = \frac{1}{2(s+p_1^2\sin^2\theta)}\left(p_1\cos\theta(m_a^2+2E_1m_b)+\sqrt{(m_a^2+2E_1m_b)^2(s+p_1^2)-4m_a^2(E_1+mb)^2(s+p_1^2\cos^2\theta)}\right) \tag{8.14}$$

In the important case of Compton scattering we have $m_a = 0$ so

$$p_* = \frac{m_b k_1}{m_b + k_1(1-\cos\theta)} \tag{8.15}$$

(we have changed the name of photon momentum to k) and the cross section takes the form:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}}^{\text{Compton}} = \frac{|T|^2}{64\pi^2}\frac{1}{[m_b+k_1(1-\cos\theta)]^2} \tag{8.16}$$

For completeness, let us present the explicit form of the Mandelstam variables for Compton scattering in the lab frame:

$$\begin{aligned} s &= m^2 + 2mk_1 \\ t &= -2mk_1\frac{k_1(1-\cos\theta)}{m+k_1(1-\cos\theta)} \\ u &= m^2 - 2mk_1\frac{m}{m+k_1(1-\cos\theta)} \end{aligned} \tag{8.17}$$

It should be mentioned that in the case when $m_a \neq 0$ but $m_a \ll m_b$ one can can still use the Compton-type formulas (8.15)-(8.17).

8.3 Dirac matrices and spinors in spinor representation

The set of Dirac matrices in the spinor representation is:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot\alpha} \\ (\bar\sigma^\mu)^{\dot\alpha\alpha} & 0 \end{pmatrix} \tag{8.18}$$

Here $\sigma^\mu = (\sigma_0, \vec\sigma)$, $\bar\sigma^\mu = (\sigma_0, -\vec\sigma)$, where σ_0 is a unit matrix and σ_i are Pauli matrices. In the explicit form:

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}, \tag{8.19}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8.20}$$

The γ_5 matrix has the form:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8.21}$$

and it anticommute with all matrices γ^μ:

$$\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu \tag{8.22}$$

Master property of γ-matrices:

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} \tag{8.23}$$

Consequences:

$$\begin{aligned}
\not{a}\ \not{a} &= a_\mu a^\mu \equiv a^2 \qquad \text{for any } 4-\text{vector a}\\
\not{a}\gamma_\mu\ \not{a} &= 2a_\mu\ \not{a} - \gamma_\mu a^2\\
\gamma_\mu\gamma_\alpha\gamma^\mu &= -2\gamma_\alpha\\
\gamma_\mu\gamma_\alpha\gamma_\beta\gamma^\mu &= 4g_{\alpha\beta}\\
\gamma_\mu\gamma_\alpha\gamma_\beta\gamma_\xi\gamma^\mu &= -2\gamma_\xi\gamma_\beta\gamma_\alpha
\end{aligned} \tag{8.24}$$

Traces:

$$\begin{aligned}
\mathrm{Tr}\,\{\gamma_\mu\gamma_\nu\} &= 4g_{\mu\nu}\\
\mathrm{Tr}\,\{\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\} &= 4(g_{\mu\nu}g_{\lambda\rho} + g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho})\\
\mathrm{Tr}\,\{\gamma_\mu\gamma_\nu\gamma_5\} &= 0\\
\mathrm{Tr}\,\{\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\gamma_5\} &= 4i\epsilon_{\mu\nu\lambda\rho}\\
\mathrm{Tr}\,\{\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\gamma_\xi\gamma_\eta\} &= 4\Big(g_{\mu\nu}(g_{\lambda\rho}g_{\xi\eta} + g_{\rho\xi}g_{\lambda\eta} - g_{\lambda\xi}g_{\rho\eta}) - g_{\mu\lambda}(g_{\nu\rho}g_{\xi\eta} + g_{\rho\xi}g_{\nu\eta} - g_{\nu\xi}g_{\rho\eta})\\
&+ g_{\mu\rho}(g_{\nu\lambda}g_{\xi\eta} - g_{\nu\xi}g_{\lambda\eta} + g_{\lambda\xi}g_{\nu\eta}) - g_{\mu\xi}(g_{\rho\eta}g_{\nu\lambda} + g_{\nu\eta}g_{\lambda\rho} - g_{\nu\rho}g_{\lambda\eta})\\
&+ g_{\mu\eta}(g_{\lambda\rho}g_{\nu\xi} + g_{\rho\xi}g_{\nu\lambda} - g_{\lambda\xi}g_{\nu\rho})\Big)
\end{aligned} \tag{8.25}$$

where ϵ is totally antisymmetric symbol ($\epsilon_{0123} = 1$). Trace of any odd number of γ-matrices is zero.

Useful formula:

$$\epsilon_{\mu\nu\alpha\beta}\epsilon^{\alpha\beta\lambda\rho} = -2\left(\delta^\lambda_\mu\delta^\rho_\nu - \delta^\lambda_\nu\delta^\rho_\mu\right) \tag{8.26}$$

Complex conjugation:

$$\gamma^\dagger_\mu = \gamma_0\gamma_\mu\gamma_0, \qquad \gamma^\dagger_5 = \gamma_5 \tag{8.27}$$

and therefore

$$\begin{aligned}
\left(\bar{u}(p)\gamma_{\mu_1}...\gamma_{\mu_n}u(p')\right)^\dagger &= \bar{u}(p')\gamma_{\mu_n}...\gamma_{\mu_1}u(p)\\
\left(\bar{v}(p)\gamma_{\mu_1}...\gamma_{\mu_n}v(p')\right)^\dagger &= \bar{v}(p')\gamma_{\mu_n}...\gamma_{\mu_1}v(p)
\end{aligned} \tag{8.28}$$

The explicit form of the spinors with definite z- component of the spin in the rest frame $\lambda = \pm\frac{1}{2}$ is:

$$\begin{aligned}
u^{(\frac{1}{2})}(p) &= \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(m + p_0 - \vec{p}\cdot\vec{\sigma})\begin{pmatrix}1\\0\end{pmatrix}\\(m + p_0 + \vec{p}\cdot\vec{\sigma})\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}\\
u^{(-\frac{1}{2})}(p) &= \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(m + p_0 - \vec{p}\cdot\vec{\sigma})\begin{pmatrix}0\\1\end{pmatrix}\\(m + p_0 + \vec{p}\cdot\vec{\sigma})\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix}
\end{aligned} \tag{8.29}$$

$$v^{(\frac{1}{2})}(p) = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix} (-m-p_0+\vec{p}\cdot\vec{\sigma})\begin{pmatrix}0\\1\end{pmatrix} \\ (m+p_0+\vec{p}\cdot\vec{\sigma})\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix}$$

$$v^{(-\frac{1}{2})}(p) = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix} (m+p_0-\vec{p}\cdot\vec{\sigma})\begin{pmatrix}1\\0\end{pmatrix} \\ (-m-p_0-\vec{p}\cdot\vec{\sigma})\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} \tag{8.30}$$

and

$$\bar{u}^{(\frac{1}{2})}(p) = \frac{1}{\sqrt{2(p_0+m)}}((1,0)(m+p_0+\vec{p}\cdot\vec{\sigma});(1,0)(m+p_0-\vec{p}\cdot\vec{\sigma}))$$

$$\bar{u}^{(-\frac{1}{2})}(p) = \frac{1}{\sqrt{2(p_0+m)}}((0,1)(m+p_0+\vec{p}\cdot\vec{\sigma});(1,0)(m+p_0-\vec{p}\cdot\vec{\sigma})) \tag{8.31}$$

$$\bar{v}^{(\frac{1}{2})}(p) = \frac{1}{\sqrt{2(p_0+m)}}((0,1)(m+p_0+\vec{p}\cdot\vec{\sigma});(0,1)(-m-p_0+\vec{p}\cdot\vec{\sigma}))$$

$$\bar{v}^{(-\frac{1}{2})}(p) = \frac{1}{\sqrt{2(p_0+m)}}((1,0)(-m-p_0-\vec{p}\cdot\vec{\sigma});(1,0)(m+p_0-\vec{p}\cdot\vec{\sigma})) \tag{8.32}$$

Here $u^{(\lambda}(p)$ and $\bar{v}^{\lambda}(p)$ are the spinors corresponding to electron and positron (respectively) with spin λ in the rest frame, and $\bar{u}^{(\lambda}(p)$,$v^{\lambda}(p)$ are the Dirac conjugate spinors.

The spinors for the states with definite helicity $h = \pm\frac{1}{2}$ are:

$$u^{[\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(m+p_0-|\vec{p}|)\omega^{(1)} \\ (m+p_0+|\vec{p}|)\omega^{(1)}\end{pmatrix}, \quad u^{[-\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(m+p_0+|\vec{p}|)\omega^{(2)} \\ (m+p_0-|\vec{p}|)\omega^{(2)}\end{pmatrix} \tag{8.33}$$

$$v^{[\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(-m-p_0-|\vec{p}|)\omega^{(2)} \\ (m+p_0-|\vec{p}|)\omega^{(2)}\end{pmatrix}, \quad v^{[-\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(m+p_0-|\vec{p}|)\omega^{(1)} \\ (-m-p_0-|\vec{p}|)\omega^{(1)}\end{pmatrix} \tag{8.34}$$

where two-component spinor ω has the form:

$$\omega^{(1)} = \begin{pmatrix} e^{-i\alpha}\cos\left(\frac{\theta}{2}\right) \\ e^{i(\phi-\alpha)}\sin\left(\frac{\theta}{2}\right)\end{pmatrix}, \qquad \omega^{(2)} = \begin{pmatrix} -e^{-i\alpha}\sin\left(\frac{\theta}{2}\right) \\ e^{i(\phi-\alpha)}\cos\left(\frac{\theta}{2}\right)\end{pmatrix} \tag{8.35}$$

where θ and ϕ are the polar and asimuthal angle of the momentum $\vec{p}$ and the phase α is arbitrary (it is convenient to choose $\alpha = \phi$, see eq. (6.153).

Let us present also the explicit form of the Dirac conjugate spinors with definite helicity:

$$\bar{u}^{[\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}}\Big(\omega^{(1)\dagger}(m+p_0+|\vec{p}|), \quad \omega^{(1)\dagger}(m+p_0-|\vec{p}|)|\Big)$$

$$\bar{u}^{[-\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}}\Big(\omega^{(2)\dagger}(m+p_0-|\vec{p}|), \quad \omega^{(2)\dagger}(m+p_0+|\vec{p}|)|\Big) \tag{8.36}$$

and

$$\bar{v}^{[\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}}\Big(\omega^{(2)\dagger}(m+p_0-|\vec{p}|), \quad \omega^{(2)\dagger}(-m-p_0-|\vec{p}|)\Big)$$

$$\bar{v}^{[-\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}}\Big(\omega^{(1)\dagger}(-m-p_0-|\vec{p}|), \quad \omega^{(1)\dagger}(m+p_0-|\vec{p}|)\Big) \tag{8.37}$$

where

$$\omega^{1\dagger} = \left(e^{i\alpha}\cos\frac{\theta}{2}, \quad e^{i(\alpha-\phi)}\sin\frac{\theta}{2}\right)$$
$$\omega^{2\dagger} = \left(-e^{i\alpha}\sin\frac{\theta}{2}, \quad e^{i(\alpha-\phi)}\cos\frac{\theta}{2}\right) \tag{8.38}$$

Properties of spinors:
1. Orthogonality

$$\begin{aligned}
\bar{u}^{\lambda}(p)u^{\lambda'}(p) = 2m\delta_{\lambda\lambda'} = -\bar{v}^{\lambda}(p)v^{\lambda'}(p) \\
\bar{u}^{\lambda}(p)\gamma^{\mu}u^{\lambda'}(p) = \bar{v}^{\lambda}(p)\gamma^{\mu}v^{\lambda'}(p) = 2p^{\mu}\delta_{\lambda\lambda'} \\
\bar{u}^{\lambda}(p)v^{\lambda'}(p) = 0 = \bar{v}^{\lambda}(p)u^{\lambda'}(p)
\end{aligned} \tag{8.39}$$

2.Completeness

$$\begin{aligned}
\textstyle\sum_{\lambda=1,2}\left(u^{\lambda}_{\alpha}(p)\bar{u}^{\lambda}_{\beta}(p) - v^{\lambda}_{\alpha}(p)\bar{v}^{\lambda}_{\beta}(p)\right) = \delta_{\alpha\beta} \\
\textstyle\sum_{\lambda=1,2} u^{\lambda}_{\alpha}(p)\bar{u}^{\lambda}_{\beta}(p) = (m+\not{p})_{\alpha\beta} \\
\textstyle\sum_{\lambda=1,2} v^{\lambda}_{\alpha}(p)\bar{v}^{\lambda}_{\beta}(p) = (\not{p}-m)_{\alpha\beta}
\end{aligned} \tag{8.40}$$

If s_μ is a four-vector of spin of the particle, then

$$\bar{u}(p,s)\gamma^{\mu}\gamma_5 u(p,s) = -\bar{v}(p,s)\gamma^{\mu}\gamma_5 v(p,s) = 2ms_{\mu} \tag{8.41}$$

and also

$$u_{\alpha}(p,s)\bar{u}_{\beta}(p,s) = \left(\frac{1+\gamma_5\not{s}}{2}(\not{p}+m)\right)_{\alpha\beta}, \quad v_{\alpha}(p,s)\bar{v}_{\beta}(p,s) = \left(\frac{1+\gamma_5\not{s}}{2}(\not{p}-m)\right)_{\alpha\beta} \tag{8.42}$$

For the particle with helicity $\frac{1}{2}$ the 4-vector of spin is $s^{\mu}(p,h=\frac{1}{2}) = (\frac{|\vec{p}|}{m}, \frac{\vec{p}p_0}{|\vec{p}|m})$ and for the particle with helicity $-\frac{1}{2}$ it is $s^{\mu}(p,h=-\frac{1}{2}) = (-\frac{|\vec{p}|}{m}, -\frac{\vec{p}p_0}{|\vec{p}|m})$

8.4 Spin of the electron

The state of the non-relativistic spin-$\frac{1}{2}$ fermion can be specified in two equivalent ways. First, one may write down this general state as a superposition of the states with spin parallel OZ and spin antiparallel OZ

$$|\Psi\rangle = \kappa^1|\uparrow\rangle + \kappa^2|\downarrow\rangle \tag{8.43}$$

and the spinor $\kappa^{\alpha} = \begin{pmatrix}\kappa^1\\ \kappa^2\end{pmatrix}$ determines the state $|\Psi\rangle$ completely. Alternatively, the state may be specifiesd by its spin. We say that the spin in the state $|\Psi\rangle$ is directed along unit vector $\vec{n}$ if this state is an eigenvector of the operator of the projection of the spin on $\vec{n}$ direction:

$$\vec{\sigma}\cdot\vec{n}|\Psi\rangle = |\Psi\rangle \tag{8.44}$$

The relation between these two descriptions is as follows. If we know the components of the spinor ξ then the spin is directed along the vector

$$\vec{n} = \kappa^\dagger \vec{\sigma} \kappa \tag{8.45}$$

Conversely, the spinor corresponding to the state with the spin pointing in the direction $\vec{n}$ is given by (8.35)

$$\kappa(\vec{n}) = \begin{pmatrix} e^{-i\alpha} \cos(\theta/2) \\ e^{i(\phi-\alpha)} \sin(\theta/2) \end{pmatrix} \tag{8.46}$$

where θ and ϕ are the polar and asimuthal angle corresponding to $\vec{n}$ and the phase α is arbitrary (it is convenient to choose $\alpha = \phi$, see eq. (6.91). It is easy to check that the spinor (8.46) satisfies the equation (8.44):

$$\vec{\sigma} \cdot \vec{n} \kappa(\vec{n}) = \kappa(\vec{n}) \tag{8.47}$$

For example, the spinor corresponding to the state with spin pointing in the x direction is $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Similarly, the relativistic spinor can be specified in two ways. First, the state of the electron moving with velocity $\vec{v}$ can be described by the spinor $\kappa^\alpha = \begin{pmatrix} \kappa^1 \\ \kappa^2 \end{pmatrix}$ in the rest frame of the electron. (In other words, to determine κ^1 and κ^2 we must **(i)** boost ourselves to the speed v so the electron will be at rest in our frame and **(ii)** measure the amplitudes of probability that the electron will have spin parallelor antipapallel OZ - this will be the components κ^1 and κ^2). The corresponding Dirac bispinor in the original frame has the form (6.112):

$$u(p) = \frac{1}{\sqrt{2(p_0 + m)}} \begin{pmatrix} (m + p_\mu \sigma^\mu) \begin{pmatrix} \kappa^1 \\ \kappa^2 \end{pmatrix} \\ (m + p_\mu \bar{\sigma}^\mu) \begin{pmatrix} \kappa^1 \\ \kappa^2 \end{pmatrix} \end{pmatrix} \tag{8.48}$$

Alternatively, one can specify the state of the relativistic electron by the four-vector of spin s_μ. This 4-vector is defined as a Lorentz boost of the vector $(0, \vec{n})$ in the rest frame (hence $s^2 = -1$ and $p \cdot s = 0$). In other words, to find the 4-vector of spin for the electron moving with velocity $\vec{v}$ we must **(i)** go to the frame where this electon is at rest, **(ii)** find the direction $\vec{n}$ where the spin is pointing, and (**iii**) mabke a boost of this vector $(0, \vec{n})$ back to the original frame. As we saw in Sect. 6.5 the relativistic generalization for the equation (8.44) has the form

$$\gamma_5 \gamma_\mu s^\mu u(p, s) = u(p, s) \tag{8.49}$$

where $\vec{p} = m\vec{v}/\sqrt{1 - v^2}$ is the momentum of the electron. Formally, the spinor with spin pointing in the 4-direction s may be defined as the eigenvector of equation (8.49).

Similarly to the non-relativistic case, one can switch back and forth between these descriptions. If we know the components of the spinor $u(p)$, then the 4-vector of spin has the form.

$$\bar{u}(p) \gamma_\mu \gamma_5 u(p) = 2m s^\mu \tag{8.50}$$

The inverse tramsition is more complicated. Suppose we know the 4-vector of spin for the electron moving with momentum p. Then in the rest frame the spin is pointing in the direction

$$\vec{n}_s = \vec{s} - \vec{p}\frac{\vec{p}\cdot\vec{s}}{|\vec{p}|^2} \tag{8.51}$$

In this frame, the corresponding non-relativistic spinor is given by eq. (8.46):

$$\kappa(\vec{n}) = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \tag{8.52}$$

where θ and ϕ are the polar and asimuthal angle corresponding to 3-vector (8.52) (we take $\alpha = 0$ for simplicity). The Dirac bispinor in the original frame is given then by eq.

$$u(p,s) \equiv \frac{1}{\sqrt{2(p_0+m)}} \begin{pmatrix} (m + p_\mu\sigma^\mu)\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \\ (m + p_\mu\bar{\sigma}^\mu)\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \end{pmatrix} \tag{8.53}$$

You may note that the expression is rather complicated. However, quite often it is not nessesary to know this explicit form of the spinor $u(p,s)$ — the set of formulas (8.41), (8.42) is sufficient for most of the practical calculations.

For completeness, let us present the corresponding formulas for positrons. In the rest frame, a positron is described by a conjugate spinor $\kappa^\dagger$. The explicit form of a spinor $\kappa^\dagger$ describing the positron with spin pointing in the direction of unit vector $\vec{n}$ is:

$$\kappa^\dagger(\vec{n}) = \big(-e^{i\phi}\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\big) \tag{8.54}$$

where θ and ϕ are the polar and asimuthal angle corresponding to $\vec{n}$. It is easy to check that the spinor (8.54) is an eigenvector of the operator of the projection of the spin on $\vec{n}$ direction:

$$\kappa^\dagger(\vec{n})(-\vec{\sigma}\cdot\vec{n}) = \kappa^\dagger(\vec{n}) \tag{8.55}$$

(Recall that the spin operator for the conjugate spinors is $-\vec{\sigma}$ since they are transformed in away opposite to usual spinors, see Eq. (6.139)).

In arbitrary frame, the spinor corresponding to 4-vector of spin $s =$(boost)$(0,\vec{n})$ takes the form:

$$\bar{v}(p,s) = \frac{1}{\sqrt{2(p_0+m)}}\big(\kappa^\dagger(\vec{n})(m + p\bar{\sigma}); \kappa^\dagger(\vec{n})(-m - p\sigma)\big) \tag{8.56}$$

which is the eigenvector of the equation

$$\bar{v}(p,s)\gamma_5 \not{s} = \bar{v}(p,s) \tag{8.57}$$

It is easy to check that in rest frame this equation reduces to (8.55). Summarising, the explicit form of the bispinor describing a positron with momentum p and 4-vector of spin s is:

$$\bar{v}(p,s) = \frac{1}{\sqrt{2(p_0+m)}}\big[\big(-e^{i\phi}\sin\tfrac{\theta}{2}, \cos\tfrac{\theta}{2}\big)(m + p\bar{\sigma}); \big(-e^{i\phi}\sin\tfrac{\theta}{2}, \cos\tfrac{\theta}{2}\big)(-m - p\sigma)\big] \tag{8.58}$$

where θ and ϕ are the polar and asimuthal angles for the vector $\vec{n}_s = \vec{s} - \vec{p}\frac{\vec{p}\cdot\vec{s}}{|\vec{p}|^2}$. The inverse formula is more simple:

$$\bar{v}(p,s)\gamma_\mu\gamma_5 v(p,s) = -2ms_\mu \tag{8.59}$$

(cf. Eq. (8.50)).

8.5 Calculation of total cross section for $MM \to MM$ scattering

The total cross section is obtained by integration of eq. (4.122) over the spherical angle:

$$\sigma_{tot} = \int d\theta \sin\theta \frac{\lambda^4}{64s\pi^2}\left(\frac{2(m^2+2|\vec{p}_1|^2)}{m^2(m^2+4|\vec{p}_1|^2)+4|\vec{p}_1|^4\sin^2(\frac{\theta}{2})} - \frac{1}{s-m^2}\right)^2 \tag{8.60}$$

Using the formulas

$$\begin{aligned}\int_{-1}^{1} dx \frac{1}{a^2-b^2x^2} &= \frac{1}{ab}\ln\frac{a+b}{a-b}\\ \int_{-1}^{1} dx \frac{1}{(a^2-b^2x^2)^2} &= -\frac{1}{2a}\frac{d}{da}\int_{-1}^{1} dx \frac{1}{(a^2-b^2x^2)} = \frac{1}{2a^3b}\ln\frac{a+b}{a-b} + \frac{1}{a^2(a^2-b^2)}\end{aligned} \tag{8.61}$$

we easily obtain

$$\begin{aligned}\int_0^{\pi} d\theta\sin\theta\frac{2(m^2+2|\vec{p}_1|^2)}{(m^2+2|\vec{p}_1|^2)^2-4|\vec{p}_1|^4\cos^2(\frac{\theta}{2})} &= \frac{1}{|\vec{p}_1|^2}\ln\frac{m^2+4|\vec{p}_1|^2}{m^2}\\ \int_0^{\pi} d\theta\sin\theta\frac{4(m^2+2|\vec{p}_1|^2)^2}{[(m^2+2|\vec{p}_1|^2)^2-4|\vec{p}_1|^4\cos^2(\frac{\theta}{2})]^2} &= \frac{1}{|\vec{p}_1|^2(2|\vec{p}_1|^2+m^2)}\ln\frac{m^2+4|\vec{p}_1|^2}{m^2} + \frac{4}{m^2(m^2+4|\vec{p}_1|^2)}\end{aligned} \tag{8.62}$$

Now it is easy to assemble the answer (4.124) from eqs. (8.60) and(8.62).

8.6 Proof of probablity conservation (4.66)

Let us check the conservation of probability (4.66) in the second order in λ. In this order the wavefunction of our state (which was a plane wave at $t = t_1$) is a five-row vector (4.63) in the Fock space:

$$\begin{aligned}\phi_M(x_2) &= (1 - iVTB)\phi_p(x_2) + \int d^3R_1 G_{(2)}(x_2 - x_1) i \overleftrightarrow{\frac{d}{dt_1}} \phi_p(x_1)\\ \phi_{M\pi}(x_2, y_2) &= \int d^3R_1 G_{(1)}(y_2, x_2; x_1) i \overleftrightarrow{\frac{d}{dt_1}} \phi_p(x_1)\\ \phi_{MMM}(x_2, x_2', x"_2) &= \int d^3R_1 G_{(2)}(x_2, x_2', x"_2; x_1) i \overleftrightarrow{\frac{d}{dt_1}} \phi_p(x_1)\\ \phi_{M\pi\pi}(x_2, y_2, y_2') &= \int d^3R_1 G_{(2)}(y_2, y_2' x_2; x_1) i \overleftrightarrow{\frac{d}{dt_1}} \phi_p(x_1)\\ \phi_{MMM\pi}(x_2, x_2', x"_2, y_2) &= G_{(1)}(x_2', x"_2, y_2)\int d^3R_1 G_0(x_2; x_1) i \overleftrightarrow{\frac{d}{dt_1}} \phi_p(x_1)\end{aligned} \tag{8.63}$$

Thus, the probability to find one M-meson at time t_2 is (cf. Eq. (4.52)):

$$P_M(t_2) = \int d^3R_2 \phi^*_M(t_2, R_2) i \overset{\leftrightarrow}{\frac{d}{dt_2}} \phi_M(t_2, R_2) \tag{8.64}$$

the probability to find M-meson and one π-meson is given by eq. (4.55) and this exhausts all the possibilities since the probability to find three particles is at best $\sim \lambda^4$ (see eq. (4.61)). So, if we sum all the probabilities in the order up to λ^2, we obtain:

$$\begin{aligned}
P_M(t_2) + P_{M\pi}(t_2) &= \delta(\vec{p} - \vec{p}') - iVTB\delta(\vec{p} - \vec{p}') + iVTB\delta(\vec{p} - \vec{p}') \\
&+ \int d^3R_1 d^3R_2 \phi^*_{p'}(x_2) i \overset{\leftrightarrow}{\frac{d}{dt_2}} G_{(2)}(x_2 - x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} \phi_p(x_1) + \int d^3R_1 d^3R_2 \phi^*_{p'}(x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} G^*_{(2)}(x_2 - x_1) i \overset{\leftrightarrow}{\frac{d}{dt_2}} \phi_p(x_2) \\
&+ \int d^3R_1 d^3R'_1 d^3R_2 d^3r_2 \phi^*_{p'}(x'_1) i \overset{\leftrightarrow}{\frac{d}{dt'_1}} G^*_{(1)}(x'_1; x_2, y_2)\, i \overset{\leftrightarrow}{\frac{d}{dx_{20}}}\, i \overset{\leftrightarrow}{\frac{d}{dy_{20}}} G_{(1)}(x_2, y_2; x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} \phi_p(x_1) \Big|_{x_{20}=y_{20}=t_2} \\
&+ \int d^3R_2 d^3R'_2 d^3R"_2 d^3r_2 \phi^*_{p'}(x'_2) G^*_{(1)}(x_2, x"_2, y_2)\, i \overset{\leftrightarrow}{\frac{d}{dx_{20}}}\, i \overset{\leftrightarrow}{\frac{d}{dx'_{20}}}\, i \overset{\leftrightarrow}{\frac{d}{dx"_{20}}}\, i \overset{\leftrightarrow}{\frac{d}{dy_{20}}} G_{(1)}(x'_2, x"_2, y_2) \phi_p(x_2) \Big|_{x_{20}=x'_{20}=x"_{20}=y_{20}=t_2}
\end{aligned} \tag{8.65}$$

It is convenient to rewrite Eq. (8.65) using

$$\phi_p(x) = \int d^3R'_1 N_0(x, x'_1) i \overset{\leftrightarrow}{\frac{d}{dt'_1}} \phi_p(x'_1), \qquad \phi^*_{p'}(x) = \int d^3R'_1 \phi^*_{p'}(x'_1) i \overset{\leftrightarrow}{\frac{d}{dt'_1}} N^*_0(x, x'_1) \tag{8.66}$$

$$\begin{aligned}
P_M(t_2) + P_{M\pi}(t_2) &= \delta(\vec{p} - \vec{p}') \\
&+ \int d^3R_1 d^3R'_1 d^3R_2 \phi^*_{p'}(x'_1) i \overset{\leftrightarrow}{\frac{d}{dt'_1}} N^*_0(x_2 - x'_1) i \overset{\leftrightarrow}{\frac{d}{dt_2}} G_{(2)}(x_2 - x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} \phi_p(x_1) \\
&+ \int d^3R_1 d^3R'_1 d^3R_2 \phi^*_{p'}(x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} G^*_{(2)}(x_2 - x_1) i \overset{\leftrightarrow}{\frac{d}{dt_2}} N_0(x_2 - x'_1) i \overset{\leftrightarrow}{\frac{d}{dt'_1}} \phi_p(x'_1) \\
&+ \int d^3R_1 d^3R'_1 d^3R_2 d^3r_2 \phi^*_{p'}(x'_1) i \overset{\leftrightarrow}{\frac{d}{dt'_1}} G^*_{(1)}(x'_1; x_2, y_2)\, i \overset{\leftrightarrow}{\frac{d}{dx_{20}}}\, i \overset{\leftrightarrow}{\frac{d}{dy_{20}}} G_{(1)}(x_2, y_2; x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} \phi_p(x_1) \Big|_{x_{20}=y_{20}=t_2} \\
&+ \int d^3R_1 d^3R'_1 d^3R_2 d^3R'_2 d^3R"_2 d^3r_2 \phi^*_{p'}(x'_1) i \overset{\leftrightarrow}{\frac{d}{dt'_1}} N^*_0(x'_2, x'_1) G^*_{(1)}(x_2, x"_2, y_2) \\
&\times\ i \overset{\leftrightarrow}{\frac{d}{dx_{20}}}\, i \overset{\leftrightarrow}{\frac{d}{dx'_{20}}}\, i \overset{\leftrightarrow}{\frac{d}{dx"_{20}}}\, i \overset{\leftrightarrow}{\frac{d}{dy_{20}}} G_{(1)}(x'_2, x"_2, y_2) N_0(x_2, x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} \phi_p(x_1) \Big|_{x_{20}=x'_{20}=x"_{20}=y_{20}=t_2}
\end{aligned} \tag{8.67}$$

The explicit expressions for $G_{(1)}$ and $G_{(2)}$ are:

$$\begin{aligned}
G_{(1)}(x_2, y_2; x_1) &= i\lambda \int d^4z_1 N_0(x_2 - z_1) G_0(y_2 - z_1) N_0(z_1 - x_1) \\
G_{(2)}(x_2 - x_1) &= -\lambda^2 \int d^4z_1 d^4z_2 N_0(x_2 - z_2) N_0(z_2 - z_1) G_0(z_2 - z_1) N_0(z_1 - x_1)
\end{aligned} \tag{8.68}$$

Substituting these eqs. into Eq. (8.69) we get

$$
\begin{aligned}
P_M(t_2)+P_{M\pi}(t_2) &= \delta\!\!\!^{-}(\vec{p}-\vec{p}') \qquad (8.69)\\
&- \lambda^2\int d^4z d^4z' d^3R_1 d^3R_1' d^3R_2 \phi^*_{p'}(x_1') i \overleftrightarrow{\frac{d}{dt_1'}} N_0^*(x_2-x_1') i \overleftrightarrow{\frac{d}{dt_2}} N_0(x_2-z')N_0(z'-z)G_0(z'-z)N_0(z-x_1) i \overleftrightarrow{\frac{d}{dt_1}} \phi_p(x_1)\\
&- \lambda^2\int d^4z d^4z' d^3R_1 d^3R_1' d^3R_2 \phi^*_{p'}(x_1) i \overleftrightarrow{\frac{d}{dt_1}} N_0^*(x_1-z')N_0^*(z'-z)G_0^*(z'-z)N_0^*(z-x_2) i \overleftrightarrow{\frac{d}{dt_2}} N_0(x_2-x_1') i \overleftrightarrow{\frac{d}{dt_1'}} \phi_p(x_1')\\
&+ \lambda^2\int d^4z d^4z' d^3R_1 d^3R_1' d^3R_2 d^3r_2 \phi^*_{p'}(x_1') i \overleftrightarrow{\frac{d}{dt_1'}} N_0^*(z'-x_1')N_0^*(x_2-z')G_0^*(y_2-z')\\
&\qquad \times\, i \overleftrightarrow{\frac{d}{dx_{20}}}\, i \overleftrightarrow{\frac{d}{dy_{20}}} N_0(x_2-z)G_0(y_2-z)N_0(z-x_1) i \overleftrightarrow{\frac{d}{dt_1}} \phi_p(x_1)\Bigg|_{x_{20}=y_{20}=t_2}
\end{aligned}
$$

where we used $G_0(x_1-x_2)=G_0(x_2-x_1)$ and similarly for N_0.

Let us consider the second term in Eq. (8.65).

$$
\begin{aligned}
&-\lambda^2\int d^3R_1 d^3R_1' d^3R_2 \phi^*_{p'}(x_1') i \overleftrightarrow{\frac{d}{dt_1'}} L_0^*(x_2-x_1') i \overleftrightarrow{\frac{d}{dt_2}} \int d^4z_1 d^4z_2 N_0(x_2-z_2)L_0(z_2-z_1)K_0(z_2-z_1)N_0(z_1-x_1) i \overleftrightarrow{\frac{d}{dt_1}} \phi_p(x_1)\\
&= -\lambda^2\int d^3R_1 d^3R_1' d^3R_2 \phi^*_{p'}(x_1') i \overleftrightarrow{\frac{d}{dt_1'}} L_0(x_1'-x_2) i \overleftrightarrow{\frac{d}{dt_2}} \int d^4z_1 d^4z_2 \Big[\theta(t_2>z_{20}>z_{10}>t_1)L_0(x_2-z_2)L_0(z_2-z_1)\\
&\quad \times K_0(z_2-z_1)L_0(z_1-x_1)+\theta(t_2>z_{10}>z_{20}>t_1)L_0(x_2-z_2)L_0(z_1-z_2)K_0(z_1-z_2)L_0(z_1-x_1)\Big] i \overleftrightarrow{\frac{d}{dt_1}} \phi_p(x_1) \qquad (8.70)
\end{aligned}
$$

(Here $\frac{d}{dt_2}$ does not act on the argument of θ-functions). Using the formula

$$
\int d^3R_2\, L_0^*(x_2-x) i \overleftrightarrow{\frac{d}{dt_2}} L_0(x_2-y) \;=\; \int d^3R_2\, L_0(x-x_2) i \overleftrightarrow{\frac{d}{dt_2}} L_0(x_2-y) \;=\; L_0(x-y) \qquad (8.71)
$$

and equations (8.66) we obtain

$$
\begin{aligned}
\text{Eq.}(8.70) &= -\lambda^2\int d^4z_1 d^4z_2\, \phi^*_{p'}(z_2)\Big[\theta(t_2>z_{20}>z_{10}>t_1)L_0(z_2-z_1) \qquad (8.72)\\
&\qquad \times K_0(z_2-z_1)+\theta(t_2>z_{10}>z_{20}>t_1)L_0(z_1-z_2)K_0(z_1-z_2)\Big]\phi_p(z_1)\\
&= -\lambda^2 \delta\!\!\!^{-}(\vec{p}-\vec{p}')\frac{1}{2E_p}\int \frac{d\!\!\!^{-}{}^3k}{4E_kE_{p-k}}\int dz_{10}dz_{20}\Big[\theta(t_2>z_{20}>z_{10}>t_1)e^{i(E_p-E_k-E_{p-k})z_{20}-i(E_p-E_k-E_{p-k})z_{10}}\\
&\qquad + \theta(t_2>z_{10}>z_{20}>t_1)e^{i(E_p+E_k+E_{p-k})z_{20}+i(E_p-E_k-E_{p-k})z_{10}}\Big]\\
&= -\lambda^2\frac{\delta\!\!\!^{-}(\vec{p}-\vec{p}')}{2E_p}\int \frac{d\!\!\!^{-}{}^3k}{4E_kE_{p-k}}\Big[\frac{it_{21}}{E_p-E_k-E_{p-k}}+\frac{1-e^{it_{21}(E_p-E_k-E_{p-k})}}{(E_p-E_k-E_{p-k})^2}+\frac{it_{21}}{E_p+E_k+E_{p-k}}+\frac{1-e^{it_{21}(E_p+E_k+E_{p-k})}}{(E_p+E_k+E_{p-k})^2}\Big]
\end{aligned}
$$

The third term in the Eq. (8.65) is the complex conjugate of the second term so

$$
\begin{aligned}
&\int d^3R_1 d^3R_2 \phi^*_{p'}(x_2) i \overleftrightarrow{\frac{d}{dt_2}} G_{(2)}(x_2-x_1) i \overleftrightarrow{\frac{d}{dt_1}} \phi_p(x_1) + \int d^3R_1 d^3R_2 \phi^*_{p'}(x_1) i \overleftrightarrow{\frac{d}{dt_1}} G^*_{(2)}(x_2-x_1) i \overleftrightarrow{\frac{d}{dt_2}} \phi_p(x_2)\\
&= -\lambda^2 \delta\!\!\!^{-}(\vec{p}-\vec{p}')\frac{1}{E_p}\int \frac{d\!\!\!^{-}{}^3k}{4E_kE_{p-k}}\Big[\frac{1-\cos(t_{21}[E_p-E_k-E_{p-k}))}{(E_p-E_k-E_{p-k}]^2}+\frac{1-\cos(it_{21}[E_p+E_k+E_{p-k}])}{(E_p+E_k+E_{p-k})^2}\Big] \qquad (8.73)
\end{aligned}
$$

Now let us turn our attention to the last term in Eq. (8.65)

$$\lambda^2 \int d^4z d^4z' d^3R_1 d^3R_1' d^3R_2 d^3r_2 \phi^*_{p'}(x_1') i \overset{\leftrightarrow}{\frac{d}{dt_1'}} L_0^*(z'-x_1') L_0^*(x_2-z') K_0^*(y_2-z')$$
$$\times \; i \overset{\leftrightarrow}{\frac{d}{dx_{20}}} i \overset{\leftrightarrow}{\frac{d}{dy_{20}}} L_0(x_2-z) K_0(y_2-z) L_0(z-x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} \phi_p(x_1)\Bigg|_{x_{20}=y_{20}=t_2} \tag{8.74}$$

Using Eq. (8.71) and Eq. (8.66) we obtain

$$\begin{aligned} \text{Eq.}(8.74) &= \lambda^2 \int_{t_1}^{t_2} dz_0 dz_0' \int d^3z d^3z' \phi^*_{p'}(z') L_0(z'-z) K_0(z'-z)\phi_p(z) \;= \qquad (8.75) \\ &= \lambda^2 \bar{\delta}(\vec{p}-\vec{p}') \frac{1}{2E_p} \int \frac{\bar{d}^3k}{4E_k E_{p-k}} \int_{t_1}^{t_2} dz_0 dz_0' e^{i(E_p-E_k-E_{p-k})z_0' - i(E_p-E_k-E_{p-k})z_0} \\ &= \lambda^2 \bar{\delta}(\vec{p}-\vec{p}') \frac{1}{E_p} \int \frac{\bar{d}^3k}{4E_k E_{p-k}} \frac{1-\cos(t_{21}[E_p-E_k-E_{p-k}])}{(E_p-E_k-E_{p-k})^2} \end{aligned}$$

The last term comes from the diagram shown in Fig. ?. It has the form

$$\int d^3R_1 d^3R_1' d^3R_2 d^3R_2' d^3R"_2 d^3r_2 \phi^*_{p'}(x_1') i \overset{\leftrightarrow}{\frac{d}{dt_1'}} N_0^*(x_2', x_1') G^*_{(1)}(x_2, x"_2, y_2)$$
$$\times \; i \overset{\leftrightarrow}{\frac{d}{dx_{20}}} i \overset{\leftrightarrow}{\frac{d}{dx'_{20}}} i \overset{\leftrightarrow}{\frac{d}{dx"_{20}}} i \overset{\leftrightarrow}{\frac{d}{dy_{20}}}\Bigg|_{x_{20}=x'_{20}=x"_{20}=y_{20}=t_2} G_{(1)}(x_2', y_2, x"_2) N_0(x_2-x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} \phi_p(x_1)$$
$$= \int d^3R_1 d^3R_1' d^3R_2 d^3R_2' d^3R"_2 d^3r_2 \int_{t_1}^{t_2} dz_0 dz_0' \int d^3z d^3z' \phi^*_{p'}(x_1') i \overset{\leftrightarrow}{\frac{d}{dt_1'}} L_0^*(x_2', x_1') L_0^*(x_2-z)$$
$$L_0^*(x"_2-z) K_0^*(y_2-z) \; i \overset{\leftrightarrow}{\frac{d}{dx_{20}}} i \overset{\leftrightarrow}{\frac{d}{dx'_{20}}} i \overset{\leftrightarrow}{\frac{d}{dx"_{20}}} i \overset{\leftrightarrow}{\frac{d}{dy_{20}}}\Bigg|_{x_{20}=x'_{20}=x"_{20}=y_{20}=t_2} L_0(x_2"-z) L_0(x_2'-z)$$
$$\times \; K_0(y_2-z) L_0(x_2-x_1) i \overset{\leftrightarrow}{\frac{d}{dt_1}} \phi_p(x_1) \tag{8.76}$$

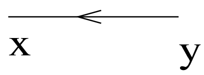
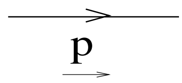
Again, using Eq. (8.71) and Eq. (8.66) we get

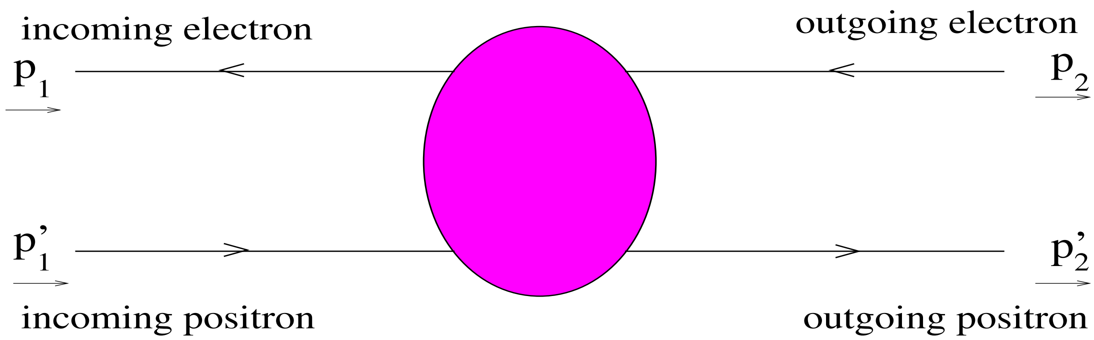
$$\begin{aligned} \text{Eq.}(8.74) &= \lambda^2 \int_{t_1}^{t_2} dz_0 dz_0' \int d^3z d^3z' \phi^*_{p'}(z) L_0(z'-z) K_0(z'-z)\phi_p(z') \;= \\ &= \lambda^2 \bar{\delta}(\vec{p}-\vec{p}') \frac{1}{2E_p} \int \frac{\bar{d}^3k}{4E_k E_{p-k}} \int_{t_1}^{t_2} dz_0 dz_0' e^{-i(E_p+E_k+E_{p-k})z_0' + i(E_p+E_k+E_{p-k})z_0} \\ &= \lambda^2 \bar{\delta}(\vec{p}-\vec{p}') \frac{1}{E_p} \int \frac{\bar{d}^3k}{4E_k E_{p-k}} \frac{1-\cos(t_{21}[E_p+E_k+E_{p-k}])}{(E_p+E_k+E_{p-k})^2} \qquad (8.77) \end{aligned}$$

One sees now that the sum of Eq. (8.73), Eq.(8.75), and Eq. (8.77) vanishes.

8.7 Arrows on the Dirac lines.

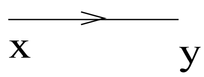
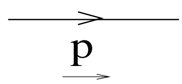
AQM lecture notes:

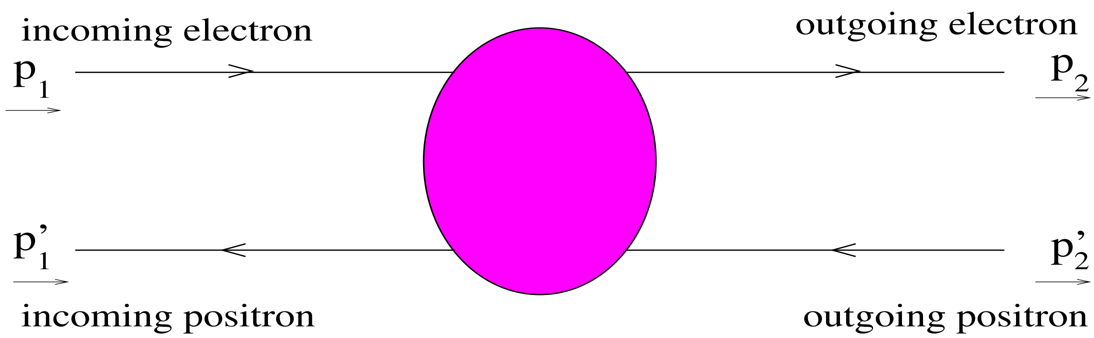
$\psi(x)\,\bar{\psi}(y) =$  $\Rightarrow$  $= \frac{m-\not{p}}{m^2-p^2}$



$$\text{arrow} \upuparrows \text{charge flow}$$

Peskin's textbook:

$\psi(x)\,\bar{\psi}(y) =$  $\Rightarrow$  $= \frac{m+\not{p}}{m^2-p^2}$



$$\text{arrow} \downuparrows \text{charge flow}$$