

1. Harmonic oscillator in classical mechanics

$$L(\varphi, \dot{\varphi}) = \frac{\dot{\varphi}^2}{2} - \frac{\omega^2}{2} \varphi^2 \quad (1)$$

$\varphi(t)$ - coordinate (usually denoted by $x(t)$)

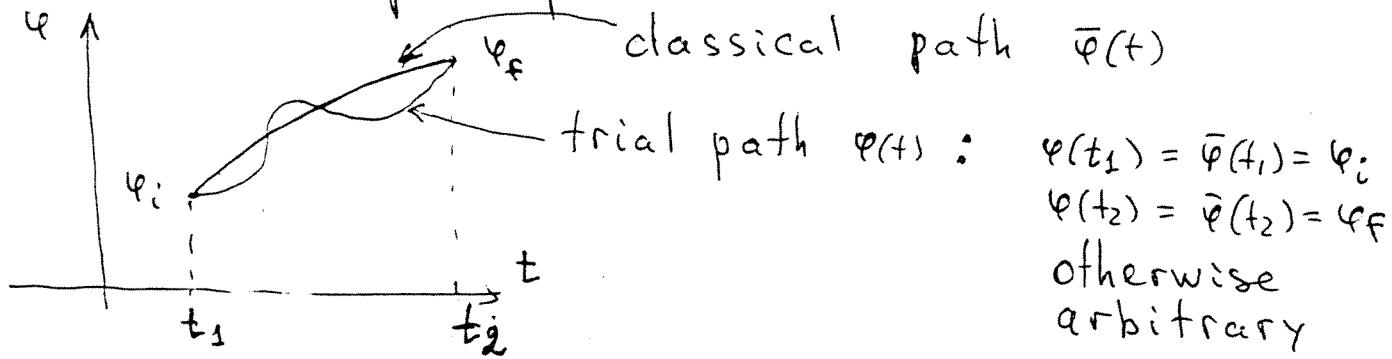
Classical eqs. of motion from least action principle

Action:

$$S(\varphi) = \int_{t_1}^{t_2} dt L(\varphi(t), \dot{\varphi}(t)) \quad (2)$$

S is a functional of $\varphi(t)$ (Mathematically,
 $S: \mathcal{L}_2 \rightarrow \mathbb{R}$)

Least action principle



$S(\varphi) \geq S(\bar{\varphi})$ - least action principle: given the initial and final points, the classical path is a path with minimal action (3)

Q: How to find a minimum of a functional?

A: Same way as for a function $F(x)$

Let x_* be a minimum of $F(x)$

Then

$$F(x_* + \Delta x) = F(x_*) + \Delta x F'(x_*) + \frac{\Delta x^2}{2} F''(x_*) + \dots \geq F(x) \quad (4)$$

$$\Rightarrow F'(x_*) = 0 \quad (5)$$

$$\text{and } F''(x_*) > 0 \quad (6)$$

We repeat the same steps for $S(\varphi)$:

Suppose $S(\bar{\varphi})$ is minimal. Therefore

$$S(\bar{\varphi} + \delta\varphi) > S(\bar{\varphi}) \quad (7)$$

for arbitrary $\delta\varphi(x)$ with boundary conditions

$$\delta\varphi(t_1) = \delta\varphi(t_2) = 0 \quad (8)$$

(recall that trial path $\varphi(t)$ has the same $\varphi(t_1)$ and $\varphi(t_2)$ as $\bar{\varphi}(t)$)

Expansion in powers of small $\delta\varphi(t)$:

$$S(\bar{\varphi} + \delta\varphi) \equiv S(\bar{\varphi}) + \int_{t_1}^{t_2} dt \delta\varphi(t) \left. \frac{\delta S}{\delta \varphi} \right|_{\varphi=\bar{\varphi}}(t) + \frac{1}{2} \int dt dt' \delta\varphi(t) \delta\varphi(t') \cdot \left. \frac{\delta^2 S}{\delta \varphi^2} \right|_{\varphi=\bar{\varphi}}(t, t') + \dots \quad (9)$$

where $\frac{\delta S}{\delta \varphi}$ (defined by eq. (9)) is called a first variational derivative of the action with respect to φ (and $\frac{\delta^2 S}{\delta \varphi^2}$ is the second functional derivative).

Similarly to the case of $F(x)$, second term in r.h.s. of eq. (9) must be 0 (and the last one must be positive):

$$\int_{t_1}^{t_2} dt \delta\varphi(t) \left. \frac{\delta S}{\delta \varphi} \right|_{\bar{\varphi}}(t) = 0 \Rightarrow \left. \frac{\delta S}{\delta \varphi} \right|_{\varphi(t)=\bar{\varphi}(t)} = 0 \quad (10)$$

because $\delta\varphi(t)$ is arbitrary.

(Second requirement means that $\frac{\delta^2 S}{\delta \varphi^2}$ is a positive-definite operator).

Let us find $\frac{\delta S}{\delta \varphi}$

$$S(\bar{\varphi} + \delta\varphi) = \int_{t_1}^{t_2} dt L(\bar{\varphi} + \delta\varphi, \dot{\bar{\varphi}} + \dot{\delta\varphi}) = \quad (11)$$

$$= \int_{t_1}^{t_2} dt \left(L(\bar{\varphi}, \dot{\bar{\varphi}}) + \delta\varphi(t) \left. \frac{\partial L(\varphi, \dot{\varphi})}{\partial \varphi} \right|_{\varphi=\bar{\varphi}} + \delta\dot{\varphi}(t) \left. \frac{\partial L}{\partial \dot{\varphi}} \right|_{\varphi=\bar{\varphi}} + O(\delta\varphi^2) \right)$$

Integration by parts

$$\int_{t_1}^{t_2} dt \left(\frac{d}{dt} \delta\varphi(t) \right) \frac{\partial L(\varphi, \dot{\varphi})}{\partial \dot{\varphi}} \Big|_{\varphi = \bar{\varphi}} = \delta\varphi(t_2) \frac{\partial L}{\partial \dot{\varphi}} \Big|_{t=t_2} - \int_{t_1}^{t_2} dt \delta\varphi(t) \frac{d}{dt} \frac{\partial L(\varphi, \dot{\varphi})}{\partial \dot{\varphi}} \Big|_{\bar{\varphi}} \quad (12)$$

because $\delta\varphi(t_1) = \delta\varphi(t_2) = 0$

$$\Rightarrow S(\bar{\varphi} + \delta\varphi) = \underbrace{\int_{t_1}^{t_2} dt L(\varphi, \dot{\varphi})}_{S(\bar{\varphi})} + \underbrace{\int_{t_1}^{t_2} dt \delta\varphi(t) \left(\frac{\partial L}{\partial \varphi} \Big|_{\bar{\varphi}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} \Big|_{\bar{\varphi}} \right)}_{\frac{\delta S}{\delta \varphi}} \quad (13)$$

$$\Rightarrow \frac{\delta S}{\delta \varphi} \Big|_{\varphi = \bar{\varphi}} = 0 \iff \frac{\partial L(\varphi, \dot{\varphi})}{\partial \varphi} \Big|_{\varphi = \bar{\varphi}} = \frac{d}{dt} \left(\frac{\partial L(\varphi, \dot{\varphi})}{\partial \dot{\varphi}} \Big|_{\varphi = \bar{\varphi}} \right) \quad (14)$$

$\frac{\partial L}{\partial \varphi} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}}$ is called the Euler-Lagrange eqn.

For the harmonic oscillator

$$L(\varphi, \dot{\varphi}) = \frac{\dot{\varphi}^2}{2} - \frac{\omega^2}{2} \varphi^2 \Rightarrow \frac{\partial L(\varphi, \dot{\varphi})}{\partial \varphi} = -\omega^2 \varphi \quad \frac{\partial L(\varphi, \dot{\varphi})}{\partial \dot{\varphi}} = \dot{\varphi} \quad (15)$$

so the Euler-Lagrange eqn takes the form

$$\frac{d}{dt} \dot{\varphi}(t) = -\omega^2 \varphi(t) \Rightarrow \ddot{\varphi}(t) = -\omega^2 \varphi(t) \quad (16)$$

↑ familiar eqn of motion for
harmonic oscillator
(solutions are $\varphi(t) = \cos \omega t, \sin \omega t$)

Hamiltonian for the harmonic oscillator
(in classical mechanics)

$$\pi(t) = \frac{\partial L}{\partial \dot{\varphi}} \quad - \text{canonical momentum} \quad (17)$$

For the harmonic oscillator

$$\pi(t) = \frac{\partial}{\partial \dot{\varphi}} L(\varphi, \dot{\varphi}) = \dot{\varphi}(t) \quad (18)$$

Classical Hamiltonian is defined as

$$H = \pi \dot{\varphi} - L(\varphi, \dot{\varphi}) \quad (19)$$

where we must express $\dot{\varphi}$ in terms of π using the equation (17)

For the harmonic oscillator we get

$$H(\pi, \varphi) = \pi \dot{\varphi} - \left(\frac{\dot{\varphi}^2}{2} - \frac{\omega^2}{2} \varphi^2 \right) \Big|_{\dot{\varphi}=\pi} = \frac{\pi^2}{2} + \frac{\omega^2}{2} \varphi^2 \quad (20)$$

2. Harmonic oscillator in quantum mechanics.

Quantization recipe:

We promote φ and π to operators $\hat{\varphi}$ and $\hat{\pi}$ satisfying the canonical commutational relation

$$[\hat{\varphi}, \hat{\pi}] = i$$

define the QM Hamiltonian

$$\hat{H} = \frac{\hat{\pi}^2}{2} + \frac{\omega^2}{2} \hat{\varphi}^2 \quad (21)$$

and solve the Schrödinger eqn

$$i \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle \quad (23)$$

where $|\psi\rangle$ is the QM vector of state.

Usually we write down the Schrödinger eqn (22) in the so-called coordinate representation of vector of state

$$\langle \psi | \psi \rangle = \psi(\varphi) \quad - \text{amplitude to discover the harmonic oscillator at the position } \varphi \quad (24)$$

$$i \frac{\partial}{\partial t} \psi(\varphi, t) = \hat{H} \psi(\varphi, t) \quad \leftarrow \text{Schrödinger eqn.} \quad (25)$$

In this representation operators $\hat{\varphi}$ and $\hat{\pi}$ are

$$\hat{\varphi} \psi(\varphi, t) = \varphi \psi(\varphi, t) \quad (26)$$

$$\hat{\pi} \psi(\varphi, t) = -i \frac{\partial}{\partial \varphi} \psi(\varphi, t)$$

(Check of commutation relation (21) :

$$[\hat{\varphi}, \hat{\pi}] \psi(\varphi) = (\hat{\varphi} \hat{\pi} - \hat{\pi} \hat{\varphi}) \psi(\varphi) = \hat{\varphi} \left(-i \frac{\partial}{\partial \varphi} \psi(\varphi) \right) - \hat{\pi} (\varphi \psi(\varphi)) = \\ = \varphi (-i) \frac{\partial}{\partial \varphi} \psi(\varphi) + i \frac{\partial}{\partial \varphi} (\varphi \psi(\varphi)) = i \psi(\varphi) \quad (27)$$

With operators $\hat{\pi}$ and $\hat{\varphi}$ in the form (26) we get the familiar expression for Schrödinger eqn

$$i \frac{\partial}{\partial t} \psi(\varphi, t) = -\frac{1}{2} \frac{\partial^2 \psi(\varphi, t)}{\partial \varphi^2} + \frac{\omega^2}{2} \varphi^2 \psi(\varphi, t) \quad (28)$$

For the stationary states

$$\psi(\varphi, t) = e^{-iEt} \psi(\varphi)$$

and therefore

$$\hat{H} \psi(\varphi) = E \psi(\varphi) \quad (29)$$

or, in the explicit form

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial \varphi^2} + \frac{\omega^2}{2} \varphi^2 \right) \psi(\varphi) = E \psi(\varphi). \quad (30)$$

Such scheme of quantization (when vector of state depends on time while the operators $\hat{\varphi}$ and $\hat{\pi}$ do not depend on t) is called the Schrödinger picture.

Heisenberg picture

$|\psi\rangle$ does not depend on time but the operators $\hat{\varphi}$ and $\hat{\pi}$ do

Construction:

take Schrödinger $|\psi\rangle$, $\hat{\pi}$, and $\hat{\varphi}$ at $t=0$

and define

$$|\psi\rangle = |\psi\rangle_{\text{Schr}}|_{t=0} \quad (31)$$

$$\hat{\varphi}(t) = e^{i\hat{H}t} \hat{\varphi} e^{-i\hat{H}t}$$

$$\hat{\pi}(t) = e^{i\hat{H}t} \hat{\pi} e^{-i\hat{H}t} \quad (32)$$

where \hat{H} is given by eq. (22) $\hat{H} = \frac{\hat{\pi}^2}{2} + \frac{\omega^2}{2} \hat{\varphi}^2$.

In this picture, instead of Schrödinger eqn. for the time evolution of the state (23) we have Heisenberg equations for the evolution of operators

$$i \frac{\partial \hat{\varphi}(t)}{\partial t} = -[\hat{H}, \hat{\varphi}(t)] \quad (33)$$

$$i \frac{\partial \hat{\pi}(t)}{\partial t} = -[\hat{H}, \hat{\pi}(t)]$$

It is instructive to check that the commutation relations (21) do not depend on time

$$[\hat{\phi}(t), \hat{\pi}(t)] = e^{i\hat{H}t} \hat{\phi} e^{-i\hat{H}t} e^{i\hat{H}t} \hat{\pi} e^{-i\hat{H}t} - (\hat{\pi} \leftrightarrow \hat{\phi}) = e^{i\hat{H}t} [\hat{\phi}, \hat{\pi}] e^{-i\hat{H}t}$$

$$= e^{i\hat{H}t} (-i) e^{-i\hat{H}t} = -i \quad (34)$$

We shall see below that the Heisenberg picture is more convenient for the description of quantum fields.

3. Classical field theory

For simplicity - scalar (Klein-Gordon) field

$$\varphi(x) \quad x: 4\text{-vector} \quad x = (t, \vec{x})$$

Lagrangian

$$L(t) = \int d^3x \overset{\uparrow}{\mathcal{L}}(x, t) \quad (35)$$

Lagrangian density

$$\mathcal{L} = \mathcal{L}(\varphi(\vec{x}, t); \partial_\mu \varphi(\vec{x}, t)) = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 \quad (36)$$

The corresponding action is

$$S(\varphi) = \int_{t_1}^{t_2} dt L(t) = \int_{t_1}^{t_2} dt \int d^3x \mathcal{L}(\varphi(\vec{x}, t); \partial_\mu \varphi(\vec{x}, t)) \quad (37)$$

Least action principle:

$$\varphi(t_1, \vec{x}) = \varphi_i(\vec{x}) \quad \text{- initial field configuration} \quad \left. \begin{array}{l} \text{fixed} \\ \varphi(t_2, \vec{x}) = \varphi_f(\vec{x}) \quad \text{- final field configuration} \end{array} \right\} \quad (38)$$

The field varies in such a way that the action $S(\bar{\varphi})$ is minimal. In other words, if we take a trial configuration $\varphi(\vec{x}, t)$ with the same boundary conditions $\varphi(\vec{x}, t_1) = \varphi_i(\vec{x})$ and $\varphi(\vec{x}, t_2) = \varphi_f(\vec{x})$ and compare its action $S(\varphi)$ to the action of the classical field configuration $S(\bar{\varphi})$

$$S(\varphi) \geq S(\bar{\varphi}) \quad (39)$$

similarly to the eq. (3) for classical mechanics.

Classical field equation from least action principle.
Similarly to the case of harmonic oscillator (see eq. 11)

$$\varphi(x) = \bar{\varphi}(x) + \delta\varphi(x) \quad (40)$$

$\begin{matrix} \text{trial} \\ \text{configuration} \end{matrix}$ $\begin{matrix} \uparrow \\ \text{small} \end{matrix}$ deviation from $\bar{\varphi}(x)$.

and

$$S(\bar{\varphi} + \delta\varphi) = S(\bar{\varphi}) + \int_{t_1}^{t_2} dt \int d^3x \delta\varphi(x) \left. \frac{\delta S(\varphi)}{\delta \varphi} \right|_{\varphi=\bar{\varphi}} + O(\delta\varphi^2) \quad (41)$$

(cf. eq. (9)).

If $S(\bar{\varphi} + \delta\varphi) \geq S(\bar{\varphi})$ the linear term in r.h.s. of eq. (41) must vanish (and the quadratic term $\sim \delta\varphi^2$ must be positive, but this is a separate issue which we will not discuss now).

$$\Rightarrow \int_{t_1}^{t_2} dt \int d^3x \delta\varphi(x) \left. \frac{\delta S}{\delta \varphi} \right|_{\varphi=\bar{\varphi}} = 0 \quad (42)$$

Since this must be true for arbitrary $\delta\varphi(t)$

$$\left. \frac{\delta S(\varphi)}{\delta \varphi} \right|_{\varphi=\bar{\varphi}} = 0 \quad \leftarrow \text{classical field equation} \quad (43)$$

Let us find the variational derivative $\frac{\delta S}{\delta \varphi}$.

$$\begin{aligned} S(\varphi + \delta\varphi) - S(\varphi) &= \int_{t_1}^{t_2} dt \int d^3x [L(\varphi + \delta\varphi, \partial_\alpha \varphi + \partial_\alpha \delta\varphi) - L(\varphi, \partial_\alpha \varphi)] \\ &= \int_{t_1}^{t_2} dt \int d^3x \left[\delta\varphi \frac{\partial L}{\partial \varphi} + \partial_\mu \delta\varphi \frac{\partial L}{\partial \partial_\mu \varphi} + O(\delta\varphi^2) \right] \end{aligned} \quad (44)$$

Consider the last term

$$\begin{aligned} &\int_{t_1}^{t_2} dt \int d^3x \left\{ \left(\frac{d}{dt} \delta\varphi(x) \right) \frac{\partial L(\varphi, \partial_\alpha \varphi)}{\partial \partial_0 \varphi} + \left(\frac{d}{dx^\mu} \delta\varphi(x) \right) \frac{\partial L(\varphi, \partial_\alpha \varphi)}{\partial \partial^\mu \varphi} \right\} = \\ &= \text{integration by parts (cf. eq. (12))} = \\ &= \cancel{\int d^3x \delta\varphi(t, \vec{x}) \frac{\partial L}{\partial \partial_0 \varphi} \Big|_{t=t_1}^{t=t_2}} - \int_{t_1}^{t_2} dt \int d^3x \delta\varphi(x) \left\{ \frac{d}{dt} \frac{\partial L(\varphi, \partial_\alpha \varphi)}{\partial \partial_0 \varphi} + \right. \\ &\quad \left. + \frac{d}{dx^\mu} \frac{\partial L(\varphi, \partial_\alpha \varphi)}{\partial \partial^\mu \varphi} \right\} \end{aligned} \quad (45)$$

Since $\delta\varphi(t_1, \vec{x}) = \delta\varphi(t_2, \vec{x}) = 0$ the first (boundary) term in r.h.s. of eq.(45) vanishes \Rightarrow

$$\Rightarrow S(\varphi + \delta\varphi) - S(\varphi) =$$

$$= \int_{t_1}^{t_2} dt \int d^3x \delta\varphi(x) \left[\frac{\partial L(\varphi, \partial_\mu \varphi)}{\partial \varphi} - \frac{d}{dx^\mu} \frac{\partial L(\varphi, \partial_\mu \varphi)}{\partial \partial^\mu \varphi} \right] \quad (46)$$

By definition, the expression in square brackets is $\frac{\delta S}{\delta \varphi}$

$$\frac{\delta S(\varphi)}{\delta \varphi} = \frac{\partial L(\varphi, \partial_\mu \varphi)}{\partial \varphi} - \frac{d}{dx^\mu} \frac{\partial L(\varphi, \partial_\mu \varphi)}{\partial \partial^\mu \varphi} \quad (47)$$

and the equation (43) for the classical field takes the familiar Euler-Lagrange form:

$$\frac{\partial L(\varphi, \partial_\mu \varphi)}{\partial \varphi} \Big|_{\varphi = \bar{\varphi}} = \frac{d}{dx^\mu} \left(\frac{\partial L(\varphi, \partial_\mu \varphi)}{\partial \partial^\mu \varphi} \Big|_{\varphi = \bar{\varphi}} \right) \quad (48)$$

For the Klein-Gordon field $L = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2$

$$\Rightarrow \frac{\partial L}{\partial \varphi} = -m^2 \varphi \quad \frac{\partial L}{\partial \partial^\mu \varphi} = \partial^\mu \varphi \quad (49)$$

and the eq.(48) takes the form

$$\frac{d}{dx^\mu} (\partial^\mu \varphi(x)) = -m^2 \varphi(x) \Leftrightarrow \partial_\mu \partial^\mu \varphi(x) = -m^2 \varphi(x) \quad (50)$$

or, in short

$$(\partial^2 + m^2) \varphi(x) = 0 \quad (51)$$

This is the form of Klein-Gordon equation which we studied in AQM course. (For now, m^2 is an arbitrary parameter in this equation, but it will be a mass of the scalar particle after quantization of the Klein-Gordon field).

Quantization of a Klein-Gordon field

For simplicity, consider the case of (1+1)-dimensional Klein-Gordon field

$$\varphi = \varphi(t, x)$$

Lagrangian for this field is

$$L(t) = \oint dx \mathcal{L}(x, t) \quad (52)$$

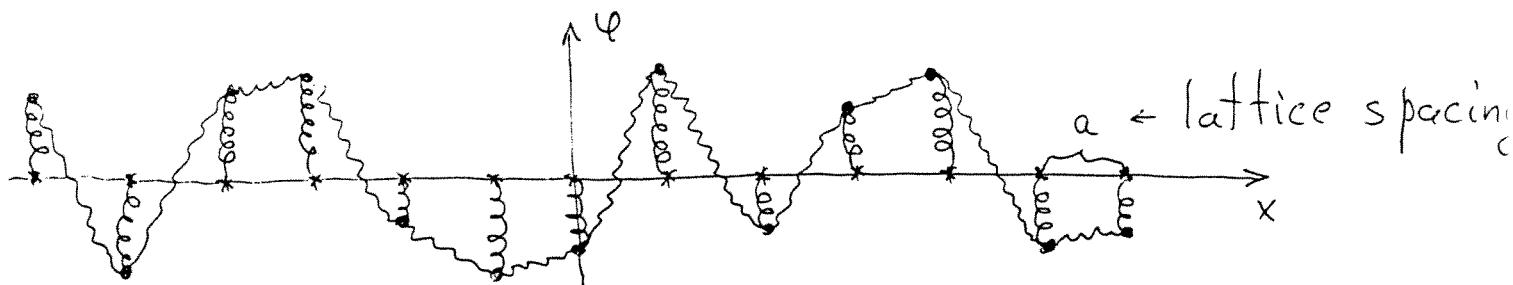
$$\mathcal{L}(x, t) = \frac{1}{2} \dot{\varphi}^2(x, t) - \frac{1}{2} \varphi'^2(x, t) - \frac{m^2}{2} \varphi^2(x, t) \quad (53)$$

$$\text{where } \varphi'(x, t) \equiv \frac{\partial \varphi(x, t)}{\partial x} \text{ (and } \dot{\varphi}(x, t) = \frac{\partial \varphi(x, t)}{\partial t})$$

Similarly to the three-dimensional case one can get the Klein-Gordon equation which has the form

$$\ddot{\varphi}(t, x) - \varphi''(t, x) = -m^2 \varphi(t, x) \quad (54)$$

Lattice model for the Klein-Gordon field:



Harmonic oscillator in each point of the lattice

$$L = \sum_{n=-\infty}^{\infty} \frac{a}{2} \dot{\varphi}_n^2(t) - \frac{m^2}{2} a \varphi_n^2(t) - \frac{1}{2a} (\varphi_{n+1}(t) - \varphi_n(t))^2 \quad (55)$$

↑
interaction between
adjacent oscillators

Let us label φ_n not by n but by the position of the lattice point $x_n = na$.

$$L(t) = \frac{a}{2} \sum_{\{x_n\}} \left[\dot{\varphi}^2(x_n, t) - \left(\frac{\varphi(x_{n+1}, t) - \varphi(x_n, t)}{a} \right)^2 - \frac{m^2}{2} \varphi^2(x_n, t) \right] \quad (55a)$$

In the "continuum limit" $a \rightarrow 0$ eq. (55a) reduces to

$$L(t) = \frac{1}{2} \int_{-\infty}^{\infty} dx (\dot{\varphi}^2(x,t) - \varphi'^2(x,t) - m^2 \varphi^2(x,t)) \leftarrow \begin{array}{l} \text{Klein-Gordon} \\ \text{Lagrangian (52)} \end{array}$$

Quantization of the set of harmonic oscillators

Canonical momenta

$$\pi_n = \frac{\partial L}{\partial \dot{\varphi}_n} = \frac{\partial}{\partial \dot{\varphi}_n} \sum_m \frac{a}{2} \dot{\varphi}_m^2 = a \dot{\varphi}_n \quad (56)$$

Let us label π_n by the point $x_n = n a$ and define

$$\pi(x_n, t) \equiv \frac{1}{a} \pi_n(t) = \dot{\varphi}_n(t) \quad (57)$$

(In the continuum limit

$$\pi(x, t) = \frac{\partial \mathcal{L}(\varphi, \dot{\varphi})}{\partial \dot{\varphi}(x, t)} \leftarrow \text{canonical momentum density} \quad (58)$$

Hamiltonian of our set of oscillators is

$$H = \sum_{n=-\infty}^{\infty} \pi_n \dot{\varphi}_n - L \Big|_{\dot{\varphi}_n = a \pi_n} = \sum \frac{\pi_n^2}{2a} + \frac{a}{2} (\varphi_{n+1} - \varphi_n)^2 + \frac{m^2}{2} \varphi_n^2 \quad (59)$$

In the notations $\varphi(x_n, t)$, $\pi(x_n, t)$ it takes the form

$$H(t) = \frac{a}{2} \sum \left\{ \pi_n^2(x_n, t) + \left(\frac{\varphi(x_n+a) - \varphi(x_n)}{a} \right)^2 + m^2 \varphi_n^2(x_n) \right\} \quad (60)$$

which reduces to

$$H(t) = \frac{1}{2} \int_{-\infty}^{\infty} dx (\pi^2(x, t) + \varphi'^2(x, t) + m^2 \varphi^2(x, t)) \quad (61)$$

in the continuum limit $a \rightarrow 0$

As usual, for quantization of a set of harmonic oscillators we promote φ_n and π_n to operators $\hat{\varphi}_n$ and $\hat{\pi}_n$ satisfying the canonical commutation relations

$$\begin{aligned} \varphi_n &\rightarrow \hat{\varphi}_n & [\hat{\varphi}_m, \hat{\pi}_n] &= i \delta_{mn} \\ \pi_n &\rightarrow \hat{\pi}_n & [\hat{\varphi}_m, \hat{\varphi}_n] &= [\hat{\pi}_m, \hat{\pi}_n] = 0 \end{aligned} \quad (62)$$

and solve the corresponding Schrödinger eqn

$$i \frac{\partial}{\partial t} |4\rangle = \hat{H} |4\rangle$$

where

(63)

$$\hat{H} = \sum_n \frac{1}{2a} \hat{\pi}_n^2 + \frac{a}{2} (\hat{\varphi}_{n+1}^2 - \hat{\varphi}_n^2) + \frac{m^2}{2} \hat{\varphi}_n^2 \quad (64)$$

and $|4\rangle$ is a vector of state of this system.

In the coordinate representation $|4\rangle$ is described by a wave function depending on all coordinates

$$|4\rangle = \psi(t; \varphi_0, \varphi_1, \varphi_{-1}, \varphi_2, \varphi_{-2}, \dots) = \psi(t; \varphi_{fin}) \quad (65)$$

probability density to observe the "0" oscillator at the position φ_0 , 1st at the position φ_1 , -1st at the position φ_{-1} etc. (at time t)

Now

$$\begin{aligned} \hat{\varphi}_n \psi(t, \varphi_{fin}) &= \varphi_n \psi(t, \varphi_{fin}) \\ \hat{\pi}_n \psi(t, \varphi_{fin}) &= -i \frac{\partial}{\partial \varphi_n} \psi(t, \varphi_{fin}) \end{aligned} \quad \left. \begin{array}{l} \text{usual} \\ \text{rules} \end{array} \right\} \quad (66)$$

It is easy to check canonical commutation relations (62)

$$\begin{aligned} [\hat{\varphi}_m, \hat{\pi}_n] \psi(t, \varphi_{fin}) &= \varphi_m (-i) \frac{\partial}{\partial \varphi_n} \psi(t, \varphi_{fin}) + i \frac{\partial}{\partial \varphi_n} (\varphi_m \psi(t, \varphi_{fin})) = \\ &= i \delta_{mn} \psi(t, \varphi_{fin}) \end{aligned} \quad (67)$$

$$[\hat{\varphi}_m, \hat{\varphi}_n] \psi(t, \varphi_{fin}) = (\varphi_m \varphi_n - \varphi_n \varphi_m) \psi(t, \varphi_{fin}) = 0 \quad (68)$$

$$[\hat{\pi}_m, \hat{\pi}_n] \psi(t, \varphi_{fin}) = i^2 \left(\frac{\partial}{\partial \varphi_m} \frac{\partial}{\partial \varphi_n} - m \leftrightarrow n \right) \psi(t, \varphi_{fin}) = 0$$

Let us now take the continuum limit $a \rightarrow 0$

The Hamiltonian is

$$\begin{aligned} \hat{H} &= \frac{a}{2} \sum_n \left[\hat{\pi}_n^2 + \left(\frac{\hat{\varphi}(x_{n+a}) - \hat{\varphi}(x_n)}{a} \right)^2 + \frac{m^2}{2} \hat{\varphi}_n^2 \right] \rightarrow \\ &\rightarrow \int dx \left\{ \frac{\hat{\pi}^2(x)}{2} + \frac{1}{2} (\hat{\varphi}'(x))^2 + \frac{m^2}{2} \hat{\varphi}^2(x) \right\} \end{aligned} \quad (69)$$

(recall that $\hat{\pi}(x_n) \equiv \frac{1}{a} \hat{\pi}_n$ by construction).

Wave function

$$|4\rangle = \psi(t; \varphi(x)) \quad (70)$$

depends now on infinite continuous set of coordinates $\varphi(x)$ (each of $\varphi(x)$ is an independent coordinate) so $\Psi(t, \varphi(x))$ is actually a wave functional.

Let us find the explicit form of the operators $\hat{\psi}(x)$ and $\hat{\pi}(x)$ in this "coordinate representation" of wave functional $\Psi(t, \varphi(x))$.

The limit $a \rightarrow 0$ for the operator $\hat{\psi}(x_n)$ is trivial!

$$\hat{\psi}(x_n) \Psi(t, \varphi(x_n)) = \varphi(x_n) \Psi(t, \varphi(x_n)) \xrightarrow{a \rightarrow 0} \hat{\psi}(x) \Psi(t, \varphi) = \varphi(x) \Psi(t, \varphi) \quad (71)$$

For the operator $\hat{\pi}(x_n)$ the limit is more tricky.

Let us prove that

$$\lim_{a \rightarrow 0} \hat{\pi}(x_n) \Psi(t, \varphi(x_n)) = \frac{\delta \Psi(t, \varphi)}{\delta \varphi(x)} \quad (72)$$

where r.h.s. is a variational derivative defined in an usual way

$$\Psi(t, \varphi(x) + h(x)) - \Psi(t, \varphi(x)) = \int dx h(x) \frac{\delta \Psi(t, \varphi)}{\delta \varphi(x)} + o(h^2(x))$$

↑
small
(we called it $\delta \varphi(x)$ before)

(73).

Consider

$$\Psi(t, \varphi(x_n) + h(x_n)) - \Psi(t, \varphi(x_n)) = \sum_{n=-\infty}^{\infty} h(x_n) \frac{\partial \Psi(t, \varphi(x_n))}{\partial \varphi(x_n)} + o(h^2) \quad (74)$$

By definition (of $\hat{\pi}(x_n)$)

$$\frac{\partial \Psi(t, \varphi(x_n))}{\partial \varphi(x_n)} = \hat{\pi}_n \Psi(t, \varphi(x_n)) = a \hat{\pi}(x_n) \Psi(t, \varphi(x_n)) \quad (75)$$

so

$$\begin{aligned} \Psi(t, \varphi(x_n) + h(x_n)) - \Psi(t, \varphi(x_n)) &= \sum_{n=-\infty}^{\infty} a h(x_n) \hat{\pi}(x_n) \Psi(t, \varphi(x_n)) \rightarrow \\ &\xrightarrow{a \rightarrow 0} \Psi(t, \varphi(x) + h(x)) - \Psi(t, \varphi(x)) = \int_{-\infty}^{\infty} dx h(x) \hat{\pi}(x) \Psi(t, \varphi(x)) \end{aligned} \quad (76)$$

Comparing this to eq. (73) we see that

$$\hat{\pi}(x) \Psi(t, \varphi) = \frac{\delta \Psi(t, \varphi)}{\delta \varphi(x)}, \quad \text{Q.E.D.} \quad (77)$$

Similarly, one can show that the limit $a \rightarrow 0$ of the canonical commutation relations

$$[\hat{\psi}(x_m), \hat{\pi}(x_n)] = +\frac{i}{a} \delta_{mn} \quad (78)$$

is

$$[\hat{\psi}(x), \hat{\pi}(y)] = i \delta(x-y) \quad (79)$$

(Proof is the same: consider

$$\sum_m a [\hat{\psi}(x_m), \hat{\pi}(x_n)] = 1 \Rightarrow$$

$$a \downarrow$$

$$\int dx [\hat{\psi}(x), \hat{\pi}(y)] = 1 \quad \text{which means that } [\hat{\psi}(x), \hat{\pi}(y)] = \delta(x-y)$$

Check:

$$\begin{aligned} [\hat{\psi}(x), \hat{\pi}(y)] \psi(t, \varphi) &= \psi(x) \left(-i \frac{\delta}{\delta \psi(y)} \right) \psi(t, \varphi) + i \frac{\delta}{\delta \psi(y)} (\psi(x) \psi(t, \varphi)) = \\ &= i \left(\frac{\delta}{\delta \psi(y)} \psi(x) \right) \psi(t, \varphi) = i \delta(x-y) \psi(t, \varphi) \end{aligned} \quad (80)$$

Reminder:

$$\begin{aligned} \psi(x) &= \int_{-\infty}^{\infty} dy \delta(x-y) \psi(y) \Rightarrow \frac{\delta}{\delta \psi(y)} \psi(x) = \frac{\delta}{\delta \psi(y)} \left(\int dy' \delta(x-y') \psi(y') \right) = \\ &= \delta(x-y) \end{aligned}$$

Now we are in a position to write down the Schrödinger equation for $\psi(t, \varphi)$:

$$i \frac{d}{dt} \psi(t, \varphi) = \frac{1}{2} \int dx \left\{ \left(\frac{\delta}{\delta \psi(x)} \right)^2 + (\psi'(x))^2 + m^2 \psi^2(x) \right\} \psi(t, \varphi) \quad (81)$$

For stationary states $\psi(t, \varphi) = e^{-iEt} \psi(\varphi)$ and the Schrödinger eqn (81) takes the form

$$\frac{1}{2} \int dx \left[\frac{\delta}{\delta \psi(x)} \frac{\delta}{\delta \psi(x)} + (\psi'(x))^2 + m^2 \psi^2(x) \right] \psi(\varphi) = E \psi(\varphi) \quad (82)$$

Real thing: $\psi(x) = \psi(\vec{x}, t)$.

Start from 3-dimensional lattice of harmonic oscillators and repeat all the steps.

Results:

$$\hat{H} = \int d^3x \left[\frac{1}{2} \hat{\pi}^2(\vec{x}) + \frac{1}{2} (\vec{\nabla} \hat{\varphi}(\vec{x}))^2 + \frac{m^2}{2} \hat{\varphi}^2(\vec{x}) \right] - \text{Hamiltonian} \quad (83)$$

Wave functional in the coordinate representation:

$$\Psi(t; \varphi(\vec{x}))$$

Operators of canonical coordinate and canonical momentum:

$$\hat{\varphi}(\vec{x}) \Psi(t; \varphi) = \varphi(\vec{x}) \Psi(t, \varphi) \quad (84)$$

$$\hat{\pi}(\vec{x}) \Psi(t; \varphi) = -i \frac{\delta}{\delta \varphi(\vec{x})} \Psi(t, \varphi)$$

$$[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{y})] = i \frac{\delta}{\delta \varphi(\vec{y})} \varphi(\vec{x}) = i \delta^{(3)}(\vec{x} - \vec{y}) \quad (85)$$

\leftarrow canonical commutation relation

Schrodinger eqn for stationary states

$$\Psi(t, \varphi) = e^{-iEt} \Psi(\varphi) \quad (86)$$

$$\hat{H} \Psi(\varphi) = E \Psi(\varphi)$$

In explicit form

$$\int d^3x \left\{ -\frac{1}{2} \left(\frac{\delta}{\delta \varphi(\vec{x})} \right)^2 + \frac{1}{2} (\vec{\nabla} \varphi(\vec{x}))^2 + \frac{m^2}{2} \varphi^2(\vec{x}) \right\} \Psi(\varphi) = E \Psi(\varphi) \quad (87)$$

Example: vacuum state.

For the harmonic oscillator

$$|0\rangle \sim e^{-\omega \varphi^2/2} \quad (88)$$

Guess: for the Klein-Gordon field the vacuum state will be similar

$$|0\rangle \sim \exp - \int d^3x \frac{1}{2} \omega \varphi^2(\vec{x})$$

But what is the analog of ω ?

For usual oscillator, the classical path is

$$\varphi(t) \sim e^{\pm i\omega t}$$

For Klein-Gordon field, the classical solution is

$$\varphi(x) \sim \int d^3 p \varphi(\vec{p}) e^{i\vec{p}\vec{x} \pm i\omega_p t} \quad (89)$$

where

$$\omega_p = \sqrt{m^2 + \vec{p}^2} \quad (90)$$

This looks like a superposition of the oscillators with ω depending on $|\vec{p}|$. \Rightarrow

Our guess for vacuum state for quantized scalar field

$$\Psi_0(\varphi) = \exp \left[-\frac{1}{2} \int d^3 x W(\varphi(\vec{x})) W(\varphi(\vec{x}')) \right] \quad (91)$$

$$\text{where } W = \sqrt{m^2 - \vec{\nabla}^2}$$

$$W(\varphi(\vec{x})) = W \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}\vec{x}} \varphi(\vec{p}) = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}\vec{x}} \omega_p \varphi(\vec{p}) \quad (92)$$

In the momentum representation it is even more simple

$$\Psi_0(\varphi) = \exp \left[-\frac{1}{2} \int d^3 p \omega_p \varphi(\vec{p}) \varphi(-\vec{p}) \right] \quad (93)$$

$$d^3 p = \frac{d^3 p}{(2\pi)^3}$$

Let us check the guess (91).

$$\hat{H} \Psi_0(\varphi) = \int d^3 x \left[-\frac{1}{2} \frac{\delta}{\delta \varphi(\vec{x})} \frac{\delta}{\delta \bar{\varphi}(\vec{x})} + \frac{1}{2} \vec{\nabla} \varphi(\vec{x}) \cdot \vec{\nabla} \bar{\varphi}(\vec{x}) + \frac{m^2}{2} \varphi^2(\vec{x}) \right] \Psi_0(\varphi) \quad (94)$$

$$\frac{\delta}{\delta \varphi(\vec{x})} \Psi_0(\varphi) = ?$$

$$\begin{aligned} \Psi_0(\varphi + \delta \varphi) &= \exp \left[-\frac{1}{2} \int d^3 x (\varphi + \delta \varphi) W(\varphi + \delta \varphi) \right] = \\ &= \exp \left[-\frac{1}{2} \int d^3 x (\varphi W \varphi + \delta \varphi W \varphi + \varphi W \delta \varphi + O(\delta \varphi)^2) \right] = \\ &= \Psi_0(\varphi) \left\{ 1 - \int d^3 x \delta \varphi(\vec{x}) W(\varphi(\vec{x})) \right\} \Rightarrow \\ &\Rightarrow \Psi_0(\varphi + \delta \varphi) - \Psi_0(\varphi) = - \int d^3 x \delta \varphi(\vec{x}) (W(\varphi(\vec{x}))) \end{aligned}$$

$$\Rightarrow \frac{\delta}{\delta \varphi(\vec{x})} \Psi_0(\varphi) = - (W(\varphi(\vec{x}))) \Psi_0(\varphi) \quad (95)$$

Similarly

$$\begin{aligned} \frac{\delta}{\delta \varphi(\vec{x})} \frac{\delta}{\delta \varphi(\vec{y})} \psi_0(\varphi) &= - \frac{\delta}{\delta \varphi(\vec{x})} (W\varphi(\vec{y})) \psi_0(\varphi) = \\ &= - \left(\frac{\delta}{\delta \varphi(\vec{x})} W\varphi(\vec{y}) \right) \psi_0(\varphi) - (W\varphi(\vec{y})) \underbrace{\frac{\delta}{\delta \varphi(\vec{x})} \psi_0(\varphi)}_{?''} \\ &\quad - W\varphi(\vec{x}) \psi_0(\varphi) \end{aligned} \quad (96)$$

Let us find the variational derivative $\frac{\delta}{\delta \varphi(\vec{x})} W\varphi(\vec{y})$

$$\begin{aligned} W\varphi(y) &= \int d^3 p \omega_p e^{i\vec{p}\vec{y}} \psi(p) = \\ &\quad " \int d^3 z e^{-i\vec{p}\vec{z}} \varphi(\vec{z}) \\ &= \int d^3 z \varphi(\vec{z}) \int d^3 p \omega_p e^{i\vec{p}(\vec{y}-\vec{z})} \end{aligned} \quad (97)$$

Therefore

$$\begin{aligned} W(\varphi(y) + \delta\varphi(y)) &= \int d^3 z (\varphi(\vec{z}) + \delta\varphi(\vec{z})) \int d^3 p \omega_p e^{i\vec{p}(\vec{y}-\vec{z})} \Rightarrow \\ \Rightarrow \frac{\delta}{\delta \varphi(\vec{x})} W\varphi(y) &= \int d^3 p \omega_p e^{i\vec{p}(\vec{y}-\vec{x})}. \end{aligned} \quad (98)$$

Finally

$$- \frac{\delta}{\delta \varphi(\vec{x})} \frac{\delta}{\delta \varphi(\vec{y})} \psi_0(\varphi) = \left[\int d^3 p \omega_p e^{i\vec{p}(\vec{y}-\vec{x})} (W\varphi)(\vec{x}) (W\varphi)(\vec{y}) \right] \psi_0(\varphi) \quad (99)$$

Now, let us take $y \rightarrow x$

$$\Rightarrow - \frac{\delta}{\delta \varphi(\vec{x})} \frac{\delta}{\delta \varphi(\vec{x})} \psi_0(\varphi) = \left[\int d^3 p \omega_p - ((W\varphi)(\vec{x}))^2 \right] \psi_0(\varphi) \quad (100)$$

$$\text{So, } \int d^3 x \left[-\frac{1}{2} \left(\frac{\delta}{\delta \varphi(\vec{x})} \right)^2 \right] \psi_0(\varphi) = \left[\int d^3 x \int d^3 p \frac{\omega_p}{2} + \int d^3 x (-\frac{1}{2}) (W\varphi(\vec{x}))^2 \right] \psi_0(\varphi) \quad (101)$$

Let us compare

$$\int d^3 x (W\varphi(\vec{x}))^2 \quad \text{and} \quad \int d^3 x (\vec{\nabla} \varphi(\vec{x}))^2$$

$$\begin{aligned}
\int d^3x (W\varphi(\vec{x}))^2 &= \int d^3x \int d^3p \omega_p e^{i\vec{p}\vec{x}} \varphi(\vec{p}) \int d^3p' \omega_{p'} e^{i\vec{p}'\vec{x}} \varphi(\vec{p}') \\
&= \int d^3p \int d^3p' \omega_p \omega_{p'} \varphi(\vec{p}) \varphi(\vec{p}') (2\pi)^3 \delta(\vec{p} + \vec{p}') = \int d^3p \omega_p^2 \varphi(\vec{p}) \varphi(-\vec{p}) \\
&= \int d^3p (m^2 + \vec{p}^2) \varphi(\vec{p}) \varphi(-\vec{p}). \tag{102}
\end{aligned}$$

$$\begin{aligned}
\int d^3x (\vec{\nabla}\varphi(\vec{x}))^2 &= \int d^3x \left(\frac{\partial}{\partial x_i} \int d^3p e^{i\vec{p}\vec{x}} \varphi(\vec{p}) \right) \left(\frac{\partial}{\partial x_i} \int d^3p' e^{i\vec{p}'\vec{x}} \varphi(\vec{p}') \right) = \\
&= \int d^3x \int d^3p \int d^3p' e^{i(\vec{p} + \vec{p}')\vec{x}} \varphi(\vec{p}) \varphi(\vec{p}') (-\vec{p} \cdot \vec{p}') = \\
&= \int d^3p \int d^3p' \varphi(\vec{p}) \varphi(\vec{p}') (-\vec{p} \cdot \vec{p}') (2\pi)^3 \delta(\vec{p} + \vec{p}') = \int d^3p \vec{p}^2 \varphi(\vec{p}) \varphi(-\vec{p}). \tag{103}
\end{aligned}$$

$$\int d^3x m^2 \varphi^2(\vec{x}) = \int d^3p m^2 \varphi(\vec{p}) \varphi(-\vec{p}) \tag{104}$$

We see that (102) = (103) + (104), so

$$\int d^3x (W\varphi(\vec{x}))^2 = \int d^3x (\vec{\nabla}\varphi(\vec{x}))^2 + m^2 \varphi^2(\vec{x}) \tag{105}$$

Thus,

$$\begin{aligned}
&\int d^3x \left[-\frac{1}{2} \left(\frac{\delta}{\delta \varphi(\vec{x})} \right)^2 + \frac{1}{2} (\vec{\nabla}\varphi(\vec{x}))^2 + \frac{m^2}{2} \varphi^2(\vec{x}) \right] \Psi_0(\varphi) = \\
&= \int d^3x \left[\int d^3p \frac{\omega_p}{2} - \frac{1}{2} (W\varphi(\vec{x}))^2 + \frac{1}{2} (\vec{\nabla}\varphi(\vec{x}))^2 + \frac{m^2}{2} \varphi^2(\vec{x}) \right] \Psi_0(\varphi) \tag{106}
\end{aligned}$$

\Rightarrow we proved that

$$\hat{H} \Psi_0(\varphi) = E_{vac} \Psi_0(\varphi) \tag{107}$$

where

$$E_{vac} = \int d^3x \int d^3p \frac{\omega_p}{2} = \overset{\text{volume of 3-space}}{V} \mathcal{E}_{vac} \tag{108}$$

and

$$\mathcal{E}_{vac} = \int d^3p \frac{\omega_p}{2} \tag{109}$$

- vacuum energy density. (Each oscillator brings $\omega_p/2$).

Quantized field: reminder

$\Psi(\varphi; t)$ - wave functional (probability for the field to be in the state $\varphi(\vec{x})$)

$\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{x})$ - operators of canonical coordinate and can. momentum

$[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})$ canonical commutation relation

Explicit form:

$$\hat{\varphi}(\vec{x}) \Psi(\varphi) = \varphi(\vec{x}) \Psi(\varphi)$$

$$\hat{\pi}(\vec{x}) \Psi(\varphi) = -i \frac{\delta}{\delta \varphi(\vec{x})} \Psi(\varphi)$$

Hamiltonian

$$\hat{H} = \int d^3x \left(\frac{1}{2} \hat{\pi}^2(\vec{x}) + \frac{1}{2} (\vec{\nabla} \hat{\varphi}(\vec{x}))^2 + \frac{m^2}{2} \hat{\varphi}^2(\vec{x}) \right)$$

Schrödinger eqn (for stationary states)

$$\hat{H}|\Psi\rangle = E|\Psi\rangle$$

In explicit form

$$\int d^3x \left(-\frac{1}{2} \left(\frac{\delta}{\delta \varphi(\vec{x})} \right)^2 + \frac{1}{2} (\vec{\nabla} \varphi(\vec{x}))^2 + \frac{m^2}{2} \varphi^2(\vec{x}) \right) \Psi(\varphi) = E \Psi(\varphi)$$

Solution with lowest energy (vacuum state)

$$\Psi_0(\varphi) = e^{-\frac{1}{2} \int d^3x \varphi(\vec{x}) W(\vec{x})} = e^{-\frac{1}{2} \int d^3p w_p \varphi(\vec{p}) \varphi(-\vec{p})}$$

$$(W \equiv \sqrt{m^2 - \vec{\nabla}^2})$$

$$\hat{H} \Psi_0(\varphi) = E_0 \Psi_0(\varphi), \quad E_0 = V \epsilon_{vac}, \quad \epsilon_{vac} = \int d^3p \frac{w_p}{2}$$

vacuum \uparrow energy density

Main problem:

$$\mathcal{H} = H + \lambda H_I \quad (\lambda \ll 1)$$

Find probabilities of transitions between different states induced by interaction (\Leftrightarrow calculate cross sections of scattering of particles, see below).

For example,

$$\lambda \hat{H}_I = \int d^3x \lambda \hat{\psi}^*(\vec{x}) \quad (111)$$

The usual procedure for construction of perturbative series would be to solve the Schrödinger equation

$$(\hat{H} + \lambda \hat{H}_I) \Psi(\varphi) = E \Psi(\varphi) \quad (112)$$

by iterations: $\Psi(\varphi) = \Psi_{(0)}(\varphi) + \lambda \Psi_{(1)}(\varphi) + \dots$

so

$$(\hat{H} + \lambda \hat{H}_I)(\Psi_{(0)} + \lambda \Psi_{(1)} + \lambda^2 \Psi_{(2)} + \dots) = (E_{(0)} + \lambda E_{(1)} + \dots)(\Psi_{(0)} + \lambda \Psi_{(1)} + \dots)$$

But: it is extremely inconvenient due

to the functional nature of $\Psi_{(0)}(\varphi), \Psi_{(1)}(\varphi)$ etc.

(Instead of usual integrals we will have the functional integrals).

Alternative approach: ladder operators formalism

For harmonic oscillator

$$\hat{a} = \frac{1}{\sqrt{2\omega}} (\omega \hat{\varphi} + i \hat{\pi}) \quad \left. \right\} \text{ladder operators} \quad (114)$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\omega}} (\omega \hat{\varphi} - i \hat{\pi}) \quad (115)$$

Commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \frac{1}{2\omega} (-i\omega [\hat{\varphi}, \hat{\pi}] + i\omega [\hat{\pi}, \hat{\varphi}]) = 1 \quad (116)$$

Hamiltonian

$$\hat{H} = \frac{\hat{\pi}^2}{2} + \frac{\omega^2}{2} \hat{\varphi}^2 = \omega (\hat{a}^\dagger \hat{a} + \frac{1}{2}) \quad (117)$$

Property:

$$\hat{a}|0\rangle = 0 \quad \hat{a} - \text{"annihilation operator"} \quad (118)$$

Proof:

$$\frac{1}{\sqrt{2\omega}} (\omega \hat{\varphi} + i \hat{\pi}) e^{-\omega \hat{a}^\dagger \hat{a}/2} = \frac{1}{\sqrt{2\omega}} (\omega \hat{\varphi} + \frac{\partial}{\partial \varphi}) e^{-\omega \hat{a}^\dagger \hat{a}/2} = 0 \quad (119)$$

The property (118) may serve as a definition of the vacuum state

Excited states:

$$|n\rangle = (\hat{a}^+)^n |0\rangle \quad E_n = \omega(n + \frac{1}{2}) \quad (120)$$

Proof

$$\begin{aligned} [\hat{H}, \hat{a}^+] &= \omega \hat{a}^+ \Rightarrow \hat{H}(\hat{a}^+)^n |0\rangle = \hat{H} \hat{a}^+ \hat{a}^+ \dots \hat{a}^+ |0\rangle \\ &= \{[\hat{H}, \hat{a}^+] (\hat{a}^+)^{n-1} + \hat{a}^+ [\hat{H}, \hat{a}^+] (\hat{a}^+)^{n-2} + \dots + (\hat{a}^+)^{n-1} [\hat{H}, \hat{a}^+] \} |0\rangle + (\hat{a}^+)^n \hat{H} |0\rangle = \\ &= \omega n (\hat{a}^+)^n |0\rangle + (\hat{a}^+)^n \frac{\omega}{2} |0\rangle = \omega(n + \frac{1}{2}) (\hat{a}^+)^n |0\rangle \Rightarrow \end{aligned} \quad (121)$$

$(\hat{a}^+)^n |0\rangle$ is an eigenvector of Hamiltonian with the eigenvalue $\omega(n + \frac{1}{2})$

$|0\rangle$ - vacuum $a^+ |0\rangle$ - first excited state
 $E_0 = \frac{\omega}{2}$ $E_{11} = \frac{\omega}{2} + \omega$

$(\hat{a}^+)^2 |0\rangle$ - second excited state, e.t.c.

$\hat{E}_{(2)} = \frac{\omega}{2} + 2\omega$
 $\Rightarrow a^+$ is called "the creation operator"

Similarly

$$[\hat{H}, \hat{a}] = -\omega \hat{a} \Rightarrow \hat{a} |n\rangle \simeq |n-1\rangle \quad (122)$$

Why these operators are better than $\hat{\varphi}$ and $\hat{\pi}$?

Example: calculation of energy shift of vacuum state due to the interaction $\hat{H}_I = \lambda \hat{\varphi}^4$

Conventional calculation ~

$$\begin{aligned} \delta E_{(0)} &\approx \langle 0 | H_I | 0 \rangle = \sqrt{\frac{\omega}{\pi}} \int d\varphi e^{-\frac{\omega\varphi^2}{2}} \lambda \varphi^4 e^{-\frac{\omega\varphi^2}{2}} = \\ &= \lambda \sqrt{\frac{\omega}{\pi}} \int_{-\infty}^{\infty} d\varphi e^{-\omega\varphi^2} \varphi^4 = \frac{3\lambda}{4\omega^2} \end{aligned} \quad (123)$$

In terms of \hat{a} and \hat{a}^+ : $\hat{\varphi} = \frac{1}{\sqrt{2\omega}} (\hat{a} + \hat{a}^+)$

$$\delta E_{(0)} = \langle 0 | \frac{\lambda}{4\omega^2} (\hat{a} + \hat{a}^+)^4 | 0 \rangle = \frac{\lambda}{4\omega^2} \langle 0 | (\hat{a} + \hat{a}^+) (\hat{a} + \hat{a}^+) (\hat{a} + \hat{a}^+) (\hat{a} + \hat{a}^+) | 0 \rangle$$

(recall that $\hat{a}|0\rangle \Rightarrow \langle 0 | \hat{a}^+ = 0$)

$$\begin{aligned} &\approx \frac{\lambda}{4\omega^2} \langle 0 | (\hat{a}^2 + [\hat{a}, \hat{a}^+] + \hat{a}^+ \hat{a}) (\hat{a}^2 + \cancel{[\hat{a}, \hat{a}^+]} + \cancel{\hat{a}^+ \hat{a} + \hat{a}^{+2}}) | 0 \rangle = \\ &= \frac{\lambda}{4\omega^2} \langle 0 | \hat{a}^2 \hat{a}^{+2} + 1 | 0 \rangle = \frac{\lambda}{4\omega^2} \langle 0 | \hat{a} ([\hat{a}, \hat{a}^+] + \hat{a}^+ \hat{a}) \hat{a}^+ + 1 | 0 \rangle \\ &= \frac{\lambda}{4\omega^2} \langle 0 | \hat{a} \hat{a}^+ + \hat{a} \hat{a}^+ \hat{a} \hat{a}^+ + 1 | 0 \rangle = \frac{\lambda}{4\omega^2} \langle 0 | [\hat{a}, \hat{a}^+] + [\hat{a}, \hat{a}^+] [\hat{a}, \hat{a}^+] + 1 | 0 \rangle = \frac{3\lambda}{4\omega^2} \end{aligned} \quad (124)$$

\Rightarrow Integration over the coordinate (x) is replaced by taking of commutators like $[\hat{a}^2, \hat{a}^{+2}]$.
For harmonic oscillator, it is about equally difficult.

But: for the quantized field we have an infinite (and worse, continuous) set of coordinates $\psi(\vec{x}) \Rightarrow$
 \Rightarrow there will be an infinite (and continuous) number of integration if we use the conventional formalism.

As to the second method, we will see in a minute that it is easily generalized for the case of a field theory.

Ladder operators for a field theory

In classical physics:

$$(\partial^2 + m^2) \psi(\vec{x}, t) = 0 \quad \text{Klein-Gordon eqn}$$

Fourier transformation

$$\psi(\vec{x}, t) = \int d^3 p e^{i\vec{p}\vec{x}} \psi(\vec{p}, t) \quad (125)$$

$$\Rightarrow (\partial^2 + m^2) \psi(\vec{x}, t) = \int d^3 p e^{i\vec{p}\vec{x}} (\frac{\partial^2}{\partial t^2} + m^2 + \vec{p}^2) \psi(\vec{p}, t) = 0$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} \psi(\vec{p}, t) + (m^2 + \vec{p}^2) \psi(\vec{p}, t) = 0$$

Each $\psi(\vec{p}, t)$ may be treated (in classical physics) as harmonic oscillator with $\omega_p = \sqrt{m^2 + \vec{p}^2}$

Guess: same will be true for the quantum field
(A quantum field $\hat{\psi}(\vec{x})$ may be described as a superposition of harmonic oscillators with $\omega_p = \sqrt{m^2 + \vec{p}^2}$).

We had:

$$\hat{\psi} = \frac{1}{\sqrt{2\omega}} (\hat{a}^+ + \hat{a}^-)$$

$$\hat{\pi} = -i\sqrt{\frac{\omega}{2}} (\hat{a}^- - \hat{a}^+)$$

We try:

$$\hat{\psi}(\vec{x}) = \int d^3 p \frac{1}{\sqrt{2\omega_p}} (\hat{a}_p e^{i\vec{p}\vec{x}} + \hat{a}_p^+ e^{-i\vec{p}\vec{x}})$$

$$\hat{\pi}(\vec{x}) = \int d^3 p (-i) \sqrt{\frac{\omega_p}{2}} (\hat{a}_p e^{i\vec{p}\vec{x}} - \hat{a}_p^+ e^{-i\vec{p}\vec{x}})$$

(126)

So,

$$\hat{a}_p = \int d^3x e^{i\vec{p}\vec{x}} \frac{1}{\sqrt{2\omega_p}} (\omega_p \hat{\varphi}(\vec{x}) + i\hat{\pi}(\vec{x})) \quad (127)$$

$$\hat{a}_p^+ = \int d^3x e^{-i\vec{p}\vec{x}} \frac{1}{\sqrt{2\omega_p}} (\omega_p \hat{\varphi}(\vec{x}) - i\hat{\pi}(\vec{x})) \quad (128)$$

Let us find $[\hat{a}_p, \hat{a}_{p'}]$.

$$\begin{aligned} [\hat{a}_p, \hat{a}_{p'}^+] &= \int d^3x d^3y e^{i\vec{p}\vec{x} - i\vec{p}'\vec{y}} \frac{1}{\sqrt{2\omega_p}} \frac{1}{\sqrt{2\omega_{p'}}} (-i\omega_p [\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{y})] + i\omega_{p'}) \\ \cdot [\hat{\pi}(\vec{x}), \hat{\varphi}(\vec{y})] &= \frac{1}{\sqrt{2\omega_p} \sqrt{2\omega_{p'}}} \int d^3x e^{i(\vec{p}-\vec{p}')\vec{x}} = i\delta^3(\vec{x}-\vec{y}) \\ &= \frac{\omega_p + \omega_{p'}}{\sqrt{2\omega_p} \sqrt{2\omega_{p'}}} (2\pi)^3 \delta(\vec{p}-\vec{p}') \Rightarrow [\hat{a}_p, \hat{a}_{p'}^+] = (2\pi)^3 \delta(\vec{p}-\vec{p}'). \end{aligned} \quad (129)$$

Similarly, it is easy to show that

$$[\hat{a}_p, \hat{a}_{p'}] = [\hat{a}_p^+, \hat{a}_{p'}^+] = 0 \quad (130)$$

Hamiltonian :

$$\begin{aligned} \hat{H} &= \frac{1}{2} \int d^3x (\hat{\pi}^2(\vec{x}) + (\vec{\nabla} \varphi(\vec{x}))^2 + m^2 \varphi^2(\vec{x})) \\ \hat{\varphi} &= \int d^3p \frac{\hat{a}_{\vec{p}} + \hat{a}_{-\vec{p}}^+}{\sqrt{2\omega_p}} e^{i\vec{p}\vec{x}} \quad \pi(\vec{x}) = \left(\int d^3p' (-i) \sqrt{\frac{\omega_{p'}}{2}} (\hat{a}_{\vec{p}'} - \hat{a}_{-\vec{p}'}^+) e^{i\vec{p}'\vec{x}} \right) \\ \Rightarrow \hat{H} &= \frac{1}{2} \int d^3x \int d^3p d^3p' e^{i(\vec{p}+\vec{p}')\vec{x}} \left\{ -\frac{\sqrt{\omega_p \omega_{p'}}}{2} (\hat{a}_{\vec{p}} - \hat{a}_{-\vec{p}}^+) (\hat{a}_{\vec{p}'} - \hat{a}_{-\vec{p}'}^+) - \right. \\ &\quad \left. - \frac{\vec{p} \cdot \vec{p}'}{2\sqrt{\omega_p \omega_{p'}}} (\hat{a}_{\vec{p}} + \hat{a}_{-\vec{p}}^+) (\hat{a}_{\vec{p}'} + \hat{a}_{-\vec{p}'}^+) + \frac{m^2}{2\sqrt{\omega_p \omega_{p'}}} (\hat{a}_{\vec{p}} + \hat{a}_{-\vec{p}}^+) (\hat{a}_{\vec{p}'} + \hat{a}_{-\vec{p}'}^+) \right\} \\ &= \frac{1}{2} \int d^3p \left\{ -\frac{\omega_p}{2} (\hat{a}_{\vec{p}} - \hat{a}_{-\vec{p}}^+) (\hat{a}_{-\vec{p}} - \hat{a}_{\vec{p}}^+) + \frac{\omega_p}{2} (\hat{a}_{\vec{p}} + \hat{a}_{-\vec{p}}^+) (\hat{a}_{-\vec{p}} + \hat{a}_{\vec{p}}^+) \right\} \\ &= \frac{1}{2} \int d^3p \frac{\omega_p}{2} \left\{ \hat{a}_{-\vec{p}}^+ \hat{a}_{-\vec{p}} + \hat{a}_{\vec{p}} \hat{a}_{\vec{p}}^+ + \hat{a}_{\vec{p}} \hat{a}_{-\vec{p}}^+ + \hat{a}_{-\vec{p}}^+ \hat{a}_{-\vec{p}} \right\} = \\ &= \int d^3p \frac{\omega_p}{2} \left\{ \hat{a}_{\vec{p}}^+ \hat{a}_{\vec{p}} + \hat{a}_{\vec{p}} \hat{a}_{\vec{p}}^+ \right\} = \int d^3p \frac{\omega_p}{2} \left\{ 2\hat{a}_{\vec{p}}^+ \hat{a}_{\vec{p}} + [\hat{a}_{\vec{p}}, \hat{a}_{\vec{p}}^+] \right\} \\ &= \int d^3p \omega_p (\hat{a}_{\vec{p}}^+ \hat{a}_{\vec{p}} + \frac{1}{2} (2\pi)^3 \delta(0)) = V \int d^3p \frac{\omega_p}{2} + \int d^3p \omega_p \hat{a}_{\vec{p}}^+ \hat{a}_{\vec{p}} \\ &\qquad \qquad \qquad \stackrel{\text{vacuum}}{\int d^3x} = V \qquad \qquad \qquad \text{vacuum energy} \\ \Rightarrow \hat{H} &= \int d^3p \omega_p \hat{a}_{\vec{p}}^+ \hat{a}_{\vec{p}} \quad \text{for our purposes} \end{aligned} \quad (131)$$

$$[\hat{H}, \hat{a}_p^+] = \int d^3 p' [\hat{a}_p^+, a_{p'}, \hat{a}_{p'}^+] \omega_{p'} = \int d^3 p' \omega_{p'} \hat{a}_{p'}^+ (2\pi)^3 \delta(\vec{p} - \vec{p}') \Rightarrow$$

$$\Rightarrow [\hat{H}, \hat{a}_p^+] = \omega_p \hat{a}_p^+$$
(132)

Similarly,

$$[\hat{H}, \hat{a}_p^-] = -\omega_p a_p^-$$

Spectrum of the Hamiltonian (131)

$$|0\rangle - \text{vacuum} \quad (\psi_0(\varphi) \sim \exp -\frac{1}{2} \int d^3 p \omega_p \varphi(\vec{p}) \varphi(-\vec{p}))$$

Property

$$\hat{a}_p |0\rangle = 0$$

a_p - "annihilation operator"

(133)

Proof

$$\begin{aligned} \hat{a}_p |0\rangle &= \int d^3 x e^{i\vec{p}\vec{x}} \frac{1}{\sqrt{2\omega_p}} (\omega_p \hat{\psi}(\vec{x}) + i\hat{\pi}(\vec{x})) \psi_0(\varphi) = \\ &= \int d^3 x e^{i\vec{p}\vec{x}} \frac{1}{\sqrt{2\omega_p}} (\omega_p \varphi(\vec{x}) + \frac{\delta}{\delta \varphi(\vec{x})}) \exp -\frac{1}{2} \int d^3 x \\ &= \int d^3 x e^{i\vec{p}\vec{x}} \frac{1}{\sqrt{2\omega_p}} (\omega_p \varphi(\vec{x}) - W \varphi(\vec{x})) \psi_0(\varphi) = 0 \quad \text{because} \\ \int d^3 x e^{i\vec{p}\vec{x}} W \varphi(\vec{x}) &= \int d^3 x e^{i\vec{p}\vec{x}} W \int d^3 p' \varphi e^{-i\vec{p}'\vec{x}} = \\ &\equiv \int d^3 x e^{i\vec{p}\vec{x}} \int d^3 p' \omega_{p'} e^{-i\vec{p}'\vec{x}} \varphi(\vec{p}') = \int d^3 p' (2\pi)^3 \delta(\vec{p} - \vec{p}') \omega_{p'} \varphi(\vec{p}') = \\ &= \omega_p \varphi(\vec{p}) = \int d^3 x e^{i\vec{p}\vec{x}} \omega_p \varphi(\vec{x}) \end{aligned}$$
(134)

Alternatively, one can define the vacuum state as a state annihilated by \hat{a}_p (eq. (133)).

In this case, one should not bother about the explicit form of the wavefunctional $\psi_0(\varphi)$: the rule $a_p |0\rangle = 0$ turns out to be sufficient for all calculations.

Next,

$$\begin{aligned} \hat{a}_p^+ |0\rangle &- \text{one-particle state} \\ \hat{H} \hat{a}_p^+ |0\rangle &= [\hat{H}, \hat{a}_p^+] |0\rangle + \hat{a}_{-(p)}^+ \underbrace{\hat{H} |0\rangle}_{=0 \text{ for our purposes}} = \omega_p \hat{a}_p^+ |0\rangle \end{aligned}$$
(135)

- $\Rightarrow \hat{a}^+(\vec{p})|0\rangle$ - eigenstate of the Hamiltonian with
 the eigenvalue $\omega_p = \sqrt{m^2 + \vec{p}^2}$. \Rightarrow
 $\Rightarrow \hat{a}^+(\vec{p})|0\rangle$ is a state with definite energy
 $E_p = \omega_p = \sqrt{m^2 + \vec{p}^2}$.

Does this state have a definite momentum \vec{p} ?

Momentum operator in a quantum field theory,
 Reminder: momentum in the classical field theory

Translation $x \rightarrow x + \epsilon$

Change in the Lagrangian $L(x + \epsilon) = L(x) + \epsilon_\mu \frac{dL}{dx_\mu}$

$$\frac{dL(\epsilon, \partial_\nu \varphi)}{dx_\mu} = \frac{\partial L}{\partial \varphi} \frac{\partial \varphi}{\partial x_\mu} + \frac{\partial L}{\partial \partial^\nu \varphi} \frac{\partial \partial_\nu \varphi}{\partial x_\mu} \quad (136)$$

From Euler - Lagrange eqs

$$\begin{aligned} \frac{\partial L}{\partial \varphi} &= \frac{d}{dx_\nu} \frac{\partial L}{\partial \partial^\nu \varphi} \Rightarrow \\ \Rightarrow \frac{dL}{dx_\mu} &= \frac{\partial \varphi}{\partial x_\mu} \frac{d}{dx_\nu} \left(\frac{\partial L}{\partial \partial^\nu \varphi} \right) + \frac{\partial^2 \varphi}{\partial x_\mu \partial x_\nu} \frac{\partial L}{\partial \partial^\nu \varphi} \Rightarrow \\ \Rightarrow \frac{dL}{dx_\mu} &= \frac{d}{dx_\nu} \left(\frac{\partial \varphi}{\partial x_\mu} \frac{\partial L}{\partial \partial^\nu \varphi} \right) \Rightarrow \underbrace{\frac{d}{dx_\nu} \left(\delta_\nu^\mu \frac{\partial L}{\partial \partial^\nu \varphi} - \delta_\nu^\mu L \right)}_{T_\nu^\mu} = 0 \end{aligned} \quad (137)$$

T_ν^μ - tensor of
 energy-momentum for
 a scalar field $\varphi(x)$

In explicit form

$$T_{\mu\nu} = \partial_\mu \varphi \frac{\partial L}{\partial \partial^\nu \varphi} - g_{\mu\nu} L = \partial_\mu \varphi \partial_\nu \varphi - \frac{g_{\mu\nu}}{2} (\partial_\alpha \varphi \partial^\alpha \varphi - m^2 \varphi^2) \quad (138)$$

Conservation of energy

$$\partial_\nu T^{00} = 0 \Rightarrow \int_{t_1}^{t_2} dt \int d^3x \left(\partial_0 T^{00} + \partial_i T^{i0} \right) = 0$$

integration by parts

$$\Rightarrow \int d^3x T^{00}(\vec{x}, t_2) = \int d^3x T^{00}(\vec{x}, t_1) \Rightarrow$$

$$\Rightarrow \int d^3x T^{00}(\vec{x}) = \int d^3x \left\{ \frac{1}{2} (\partial_0 \varphi)^2 + \frac{1}{2} (\vec{\nabla} \varphi)^2 + \frac{m^2}{2} \varphi^2 \right\} = \text{const} \quad (139)$$

Since $\partial_0 \varphi(\vec{x}, t) = \pi(\vec{x}, t)$

$$\int d^3x T^{00}(\vec{x}, t) = \underbrace{\int d^3x \left\{ \frac{1}{2} \pi^2(\vec{x}, t) + \frac{1}{2} \vec{\nabla} \varphi(\vec{x}, t) + \frac{m^2}{2} \varphi(\vec{x}, t) \right\}}_{\text{classical Hamiltonian}} = \text{const} \quad (14)$$

(14)
classical Hamiltonian \Leftrightarrow energy of the field

Conservation of the momentum

$$\partial_\nu T^{\nu i} = 0 \Rightarrow \int_{t_1}^{t_2} dt \int d^3x (\partial_0 T^{0i} + \cancel{\partial_\nu T^{\nu i}}) = 0 \quad (14)$$

integration by parts

$$\Rightarrow \int d^3x T^{0i}(\vec{x}, t_2) = \int d^3x T^{0i}(\vec{x}, t_1) \Rightarrow$$

$$\Rightarrow \int d^3x T^{0i}(\vec{x}, t) = \int d^3x \partial^0 \varphi(\vec{x}, t) \partial^i \varphi(\vec{x}, t) = \int d^3x \pi(\vec{x}, t) \partial^i \varphi(\vec{x}, t) = \text{const} \quad (142)$$

$P^i = \int d^3x \pi(\vec{x}, t) \partial^i \varphi(\vec{x}, t)$ is the (conserved) classical momentum of the field $\varphi(\vec{x}, t)$

Corresponding quantum operator is

$$\hat{P}_i = \int d^3x \hat{\pi}(\vec{x}) \partial_i \hat{\varphi}(\vec{x}) \quad (143)$$

In terms of ladder operators

$$\begin{aligned} \hat{P}_i &= \int d^3x \int d^3p (-i) \sqrt{\frac{\omega_p}{2}} e^{i \vec{p} \cdot \vec{x}} (\hat{a}_p - \hat{a}_{-p}^\dagger) \int d^3p' i p'_i \frac{1}{\sqrt{2\omega_{p'}}} e^{i \vec{p}' \cdot \vec{x}} (\hat{a}_{p'}^\dagger + \hat{a}_{-p'}^\dagger) = \\ &= \int d^3p \frac{1}{2} (\hat{a}_p - \hat{a}_{-p}^\dagger) (\hat{a}_{-p} + \hat{a}_p^\dagger) p_i = \frac{1}{2} \int d^3p p_i (\hat{a}_p \hat{a}_{-p} - \hat{a}_{-p}^\dagger \hat{a}_{-p} + \hat{a}_p^\dagger \hat{a}_p - \hat{a}_{-p}^\dagger \hat{a}_p^\dagger) = \boxed{(p \leftrightarrow -p)} \Rightarrow \text{first and last terms vanish} \\ &= \frac{1}{2} \int d^3p p_i (\hat{a}_p \hat{a}_p^\dagger + \hat{a}_p^\dagger \hat{a}_p) = \int d^3p p_i (\hat{a}_p^\dagger a_p + \frac{1}{2}) \end{aligned} \quad (144)$$

NB: $\int d^3p p_i$ vanish (due to rotational invariance, there is no preferred direction of "vacuum momentum")

$$\hat{P}_i |0\rangle = \frac{1}{2} \int d^3p p_i |0\rangle = \text{should be zero}$$

Finally $\hat{P}_i = \int d^3p p_i \hat{a}_p^\dagger \hat{a}_p$ - momentum operator (145)

(cf. $\hat{P}_0 \equiv \hat{H} = \int d^3p \omega_p \hat{a}_p^\dagger \hat{a}_p$ - energy operator (Hamiltonian))

Commutators:

$$[\hat{P}_i, \hat{a}_{\vec{p}}^+] = \oint d^3 p' p'_i \omega_{p'} [\hat{a}_{\vec{p}}^+, \hat{a}_{\vec{p}'}^+], \quad [\hat{a}_{\vec{p}}^+, \hat{a}_{\vec{p}'}^+] = \oint d^3 p' p'_i \omega_{p'} \hat{a}_{\vec{p}'}^+ [\hat{a}_{\vec{p}}^+, \hat{a}_{\vec{p}'}^+] \\ = \oint d^3 p' p'_i \hat{a}_{\vec{p}'}^+ (2\pi)^3 \delta(\vec{p} - \vec{p}') = p_i \hat{a}_{\vec{p}}^+$$
(146)

$$[\hat{P}_i, \hat{a}_{\vec{p}}^+] = -p_i \hat{a}_{\vec{p}}^+ \quad \text{similarly}$$
(147)

Now we are in a position to check that the state $\hat{a}_{\vec{p}}^+ |0\rangle$ has a definite momentum \vec{p} .

$$\hat{P}_i \hat{a}_{\vec{p}}^+ |0\rangle = \hat{a}_{\vec{p}}^+ \cancel{\hat{P}_i} |0\rangle + [\hat{P}_i, \hat{a}_{\vec{p}}^+] |0\rangle = p_i \hat{a}_{\vec{p}}^+ |0\rangle$$
(148)

^{o (no "vacuum momentum")}

$\Rightarrow \hat{a}_{\vec{p}}^+ |0\rangle$ is an eigenstate of momentum operator \hat{P}_i with eigenvalue $p_i \Leftrightarrow \hat{a}_{\vec{p}}^+ |0\rangle$ is a state with momentum \vec{p}

We interpret $\hat{a}_{\vec{p}}^+ |0\rangle$ as the one-particle state with momentum \vec{p} and energy $E = \omega_p = \sqrt{m^2 + \vec{p}^2}$ (recall that $\hat{H} \hat{a}_{\vec{p}}^+ |0\rangle = \omega_p \hat{a}_{\vec{p}}^+ |0\rangle$)

Two-particle state: $\hat{a}_{\vec{p}}^+ \hat{a}_{\vec{q}}^+ |0\rangle$

Check:

$$\hat{P}_i \hat{a}_{\vec{p}}^+ \hat{a}_{\vec{q}}^+ |0\rangle = [\hat{P}_i, \hat{a}_{\vec{p}}^+] \hat{a}_{\vec{q}}^+ |0\rangle + \hat{a}_{\vec{p}}^+ [\hat{P}_i, \hat{a}_{\vec{q}}^+] |0\rangle + \cancel{\hat{a}_{\vec{p}}^+ \hat{a}_{\vec{q}}^+ \hat{P}_i |0\rangle} \\ = p_i \hat{a}_{\vec{p}}^+ \hat{a}_{\vec{q}}^+ |0\rangle + a_{\vec{p}q_i}^+ \hat{a}_{\vec{q}}^+ |0\rangle = (p_i + q_i) \hat{a}_{\vec{p}}^+ \hat{a}_{\vec{q}}^+ |0\rangle$$
(149)

$$\hat{H} \hat{a}_{\vec{p}}^+ \hat{a}_{\vec{q}}^+ |0\rangle = \text{same trick} = (\omega_p + \omega_q) \hat{a}_{\vec{p}}^+ \hat{a}_{\vec{q}}^+ |0\rangle$$
(150)

$\Rightarrow \hat{a}_{\vec{p}}^+ \hat{a}_{\vec{q}}^+ |0\rangle$ is a state with momentum $\vec{p} + \vec{q}$ and energy $E_p + E_q \Rightarrow$ it is a two-particle state

Note that $\hat{a}_{\vec{p}}^+ \hat{a}_{\vec{q}}^+ |0\rangle = \hat{a}_{\vec{q}}^+ \hat{a}_{\vec{p}}^+ |0\rangle \Rightarrow$ Bose-Einstein statistics

n-particle state:

$\hat{a}_{\vec{p}_1}^+ \dots \hat{a}_{\vec{p}_n}^+ |0\rangle$ = state with n particles with momenta $\vec{p}_1, \vec{p}_2, \dots, \vec{p}_n$

Heisenberg picture in QFT.

In quantum mechanics

Schrödinger picture

Dynamics:

$$\begin{aligned} \Psi(t) & \text{- vector of state depends on time} \\ \hat{\pi} & \text{- can. momentum} \\ \hat{\varphi} & \text{- can. coordinate} \end{aligned} \quad \left. \begin{array}{l} \text{do not depend on } t \\ \} \end{array} \right.$$

$$i \frac{\partial \Psi(t)}{\partial t} = \hat{H} \Psi(t) \quad \hat{H} = \frac{\hat{\pi}^2}{2} + \frac{\omega^2}{2} \hat{\varphi}^2 \quad \text{- Schrödinger eqn.}$$

Heisenberg picture

Dynamics

$$\begin{aligned} \Psi &= \Psi_{\text{Schrö}}(t=0) \text{ - time-independent vector of state} \\ \hat{\pi}(t) &= e^{i\hat{H}t} \hat{\pi} e^{-i\hat{H}t} \\ \hat{\varphi}(t) &= e^{i\hat{H}t} \hat{\varphi} e^{-i\hat{H}t} \end{aligned} \quad \left. \begin{array}{l} \text{depend on time} \\ \} \end{array} \right.$$

$$\frac{\partial \hat{\varphi}}{\partial t} = i [\hat{H}, \hat{\varphi}] ; \quad \frac{\partial \hat{\pi}}{\partial t} = i [\hat{H}, \hat{\pi}] \quad \text{- Heisenberg eqs.}$$

In QFT: $\hat{\pi}(\vec{x}), \hat{\varphi}(\vec{x}), \Psi(t, \varphi(\vec{x}))$ - Schrödinger picture

$$i \frac{\partial}{\partial t} \Psi(t, \varphi(\vec{x})) = \int d^3x \left(-\frac{1}{2} \frac{\delta^2}{\delta \varphi(\vec{x})^2} + \frac{(\vec{\nabla} \varphi(\vec{x}))^2}{2} + \frac{m^2}{2} \varphi^2 \right) \Psi(t, \varphi(\vec{x})) \quad \leftarrow \text{dynamics (Schrö eqn)}$$

Transition to Heisenberg picture -
- same as in quantum mechanics

Define $\Psi(\varphi(\vec{x})) \equiv \Psi_{\text{Schrö}}(0, \varphi(\vec{x}))$ - vector of state

(For example, vacuum state
is $\Psi_0(\varphi) = \exp -\frac{1}{2} \int d^3x \varphi(\vec{x}) W \varphi(\vec{x})$)

$$\begin{aligned} \hat{\varphi}(\vec{x}) &\equiv \hat{\varphi}(\vec{x}, t) \equiv e^{i\hat{H}t} \hat{\varphi}(\vec{x}) e^{-i\hat{H}t} \\ \hat{\pi}(\vec{x}) &\equiv \hat{\pi}(\vec{x}, t) \equiv e^{i\hat{H}t} \hat{\pi}(\vec{x}) e^{-i\hat{H}t} \end{aligned} \quad (151)$$

Dynamics will be governed by Heisenberg equations:

$$\frac{\partial}{\partial t} \hat{\varphi}(\vec{x}, t) = i [\hat{H}, \hat{\varphi}(\vec{x}, t)] , \quad \frac{\partial}{\partial t} \hat{\pi}(\vec{x}, t) = i [\hat{H}, \hat{\pi}(\vec{x}, t)] \quad (152)$$

Instead of variational derivatives in Schrödinger eqn
we have simple derivatives in Heisenberg eqs.

In terms of ladder operators

$$\hat{\psi}(\vec{x}, t) = \int d^3 p \frac{1}{\sqrt{2E_p}} (\hat{a}_p e^{-ipx} + \hat{a}_p^+ e^{ipx}) \Big|_{p_0 = E_p \equiv \omega_p = \sqrt{m^2 + \vec{p}^2}} \quad (153)$$

$$\hat{\pi}(\vec{x}, t) = \int d^3 p (-i) \sqrt{\frac{E_p}{2}} (\hat{a}_p e^{-ipx} - \hat{a}_p^+ e^{ipx}) \Big|_{p_0 = E_p} \quad (154)$$

Proof:

Note that

$$\begin{aligned} e^{i\hat{H}t} \hat{a}_p e^{-i\hat{H}t} &= \sum_{n=0}^{\infty} i^n \frac{t^n}{n!} [\underbrace{\hat{H}[\hat{H} \dots [\hat{H}}_n, \hat{a}_p]]] = \\ &= \sum_{n=0}^{\infty} i^n \frac{t^n}{n!} (-E_p)^n \hat{a}_p = \hat{a}_p e^{-iE_pt} \end{aligned} \quad (155)$$

(recall that $[\hat{H}, \hat{a}_p] = -E_p \hat{a}_p$)

Similarly

$$e^{i\hat{H}t} \hat{a}_p^+ e^{-i\hat{H}t} = \hat{a}_p^+ e^{iE_pt} \quad (\text{recall that } [\hat{H}, \hat{a}_p^+] = E_p \hat{a}_p^+) \quad (156)$$

$$\begin{aligned} \Rightarrow e^{i\hat{H}t} \hat{\psi}(\vec{x}) e^{-i\hat{H}t} &= e^{i\hat{H}t} \int d^3 p \frac{1}{\sqrt{2E_p}} (\hat{a}_p e^{i\vec{p}\vec{x}} + \hat{a}_p^+ e^{-i\vec{p}\vec{x}}) e^{-i\hat{H}t} = \\ &= \int d^3 p \frac{1}{\sqrt{2E_p}} \left(\underbrace{e^{i\hat{H}t} \hat{a}_p e^{-i\hat{H}t}}_{\hat{a}_p e^{-iE_pt}} e^{i\vec{p}\vec{x}} + \underbrace{e^{i\hat{H}t} \hat{a}_p^+ e^{-i\hat{H}t}}_{\hat{a}_p^+ e^{iE_pt}} e^{-i\vec{p}\vec{x}} \right) = \\ &= \int d^3 p \frac{1}{\sqrt{2E_p}} (\hat{a}_p e^{-iE_pt + i\vec{p}\vec{x}} + \hat{a}_p^+ e^{iE_pt - i\vec{p}\vec{x}}) \Rightarrow \hat{\psi}(\vec{x}) = (153) \end{aligned}$$

Also

$$\begin{aligned} e^{i\hat{H}t} \hat{\pi}(\vec{x}) e^{-i\hat{H}t} &= \int d^3 p (-i) \sqrt{\frac{E_p}{2}} (e^{i\hat{H}t} \hat{a}_p e^{-i\hat{H}t} e^{i\vec{p}\vec{x}} - \\ &- e^{i\hat{H}t} \hat{a}_p^+ e^{-i\hat{H}t} e^{-i\vec{p}\vec{x}}) = \int d^3 p (-i) \sqrt{\frac{E_p}{2}} (\hat{a}_p e^{-iE_pt + i\vec{p}\vec{x}} - \hat{a}_p^+ e^{iE_pt - i\vec{p}\vec{x}}) \end{aligned}$$

$$\Rightarrow \hat{\pi}(\vec{x}) = (154)$$



It is easy to see that

$$\frac{\partial}{\partial t} \hat{\psi}(\vec{x}, t) = \int d^3 p \frac{1}{\sqrt{2E_p}} (-iE_p \hat{a}_p e^{-ipx} + iE_p \hat{a}_p^+ e^{ipx}) = \hat{\pi}(\vec{x}, t) \quad (157)$$

$$\begin{aligned} \frac{\partial}{\partial t} \hat{\pi}(\vec{x}, t) &= \int d^3 p (-i\sqrt{\frac{E_p}{2}}) (-iE_p) (\hat{a}_p e^{-ipx} + \hat{a}_p^+ e^{ipx}) = \\ &= \int d^3 p \frac{-E_p^2}{\sqrt{2E_p}} (\hat{a}_p e^{-ipx} + \hat{a}_p^+ e^{ipx}) = - \int d^3 p (m^2 + \vec{p}^2) \frac{1}{\sqrt{2E_p}} (\hat{a}_p e^{-ipx} + \\ &+ \hat{a}_p^+ e^{ipx}) = (-m^2 + \vec{p}^2) \int d^3 p \frac{1}{\sqrt{2E_p}} (\hat{a}_p e^{-ipx} + \hat{a}_p^+ e^{ipx}) = (-m^2 + \vec{p}^2) \hat{\psi}(\vec{x}) \end{aligned}$$

$$\Rightarrow \left(\frac{\partial}{\partial t}\right)^2 \hat{\varphi}(\vec{x}, t) = \frac{\partial}{\partial t} \hat{\pi}(\vec{x}, t) = (-m^2 + \vec{\nabla}^2) \varphi(\vec{x}, t) \Rightarrow \quad (158)$$

$$\Rightarrow (\partial^2 + m^2) \varphi(\vec{x}, t) = 0 \quad \leftarrow \text{Klein-Gordon eqn same as for classical field } \varphi(\vec{x}, t).$$

We defined

$$\varphi(\vec{x}, t) = e^{i\hat{H}t} \varphi(\vec{x}) e^{-i\hat{H}t}$$

One can also prove that

$$\varphi(\vec{x}) = e^{-i\hat{P}\vec{x}} \varphi(0) e^{i\hat{P}\vec{x}} \quad (159)$$

where \hat{P} is the momentum operator (145)

Proof: similar to eq. (155)

Note that

$$\begin{aligned} e^{-i\hat{P}\vec{x}} \hat{a}_p e^{i\hat{P}\vec{x}} &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \underbrace{[\hat{P}\vec{x} [\hat{P}\vec{x} [\hat{P}\vec{x}, \dots [\hat{P}\vec{x}, \hat{a}_p]]]]}_{n} = \\ [\hat{P}\vec{x}, \hat{a}_p] &= x_i [\hat{p}_i, \hat{a}_p] = -x_i p_i \hat{a}_p = -\vec{p}\vec{x} a_p \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} (-\vec{p}\vec{x})^n a_p = a_p e^{i\vec{p}\vec{x}} \end{aligned} \quad (160)$$

$$\text{Similarly } [\hat{P}\vec{x}, \hat{a}_p^+] = \vec{p}\vec{x} a_p^+ \Rightarrow e^{-i\hat{P}\vec{x}} \hat{a}_p^+ e^{i\hat{P}\vec{x}} = \hat{a}_p^+ e^{-i\vec{p}\vec{x}} \quad (161)$$

$$\begin{aligned} e^{-i\hat{P}\vec{x}} \varphi(0) e^{i\hat{P}\vec{x}} &= e^{-i\hat{P}\vec{x}} \int \frac{d^3 p}{\sqrt{2E_p}} (\hat{a}_p + \hat{a}_p^+) e^{i\hat{P}\vec{x}} = \\ &= \int \frac{d^3 p}{\sqrt{2E_p}} (e^{-i\hat{P}\vec{x}} \hat{a}_p e^{i\hat{P}\vec{x}} + e^{-i\hat{P}\vec{x}} \hat{a}_p^+ e^{i\hat{P}\vec{x}}) = \\ &= \int \frac{d^3 p}{\sqrt{2E_p}} (\hat{a}_p e^{i\vec{p}\vec{x}} + \hat{a}_p^+ e^{-i\vec{p}\vec{x}}) = \varphi(\vec{x}). \end{aligned}$$

Combining eqs. (159) and (161) one gets

$$\varphi(\vec{x}, t) = e^{i\hat{H}t} \varphi(\vec{x}) e^{-i\hat{H}t} = e^{i\hat{H}t} e^{-i\hat{P}\vec{x}} \varphi(0) e^{i\hat{P}\vec{x}} e^{-i\hat{H}t} \quad (162)$$

Operators \hat{H} and \hat{P} commute

$$\begin{aligned} [\hat{H}, \hat{P}] &= \int d^3 p d^3 k w_p \vec{k} [\hat{a}_p^+ \hat{a}_p, \hat{a}_k^+ \hat{a}_k] = \int d^3 p d^3 k w_p \vec{k} \cdot \\ \cdot (\hat{a}_p^+ \hat{a}_k [\hat{a}_p, \hat{a}_k^+] + \hat{a}_p \hat{a}_k^+ [\hat{a}_p^+, \hat{a}_k]) &= \int d^3 p d^3 k w_p \vec{k} \delta(\vec{p} - \vec{k}) (2\pi)^3. \\ \cdot (\hat{a}_p^+ \hat{a}_k - \hat{a}_p \hat{a}_k^+) &= \int d^3 p w_p \vec{p} [\hat{a}_p, \hat{a}_p^+] \sim \int d^3 p w_p \vec{p} = 0 \end{aligned} \quad (163)$$

$$[\hat{H}, \hat{\vec{P}}] = 0 \Rightarrow e^{i\hat{H}t} e^{-i\hat{\vec{P}}\vec{x}} = e^{i\hat{H}t - i\hat{\vec{P}}\vec{x}} = e^{i\hat{P}\vec{x}} \quad (164)$$

where $\hat{P} = (\hat{H}, \hat{\vec{P}})$ is the operator of the 4-momentum

$$\hat{P}_\mu = \int d^3p P_\mu a_p^\dagger a_p \quad (165)$$

(for operator $\hat{P}_0 \equiv H$ we must take $p_0 = \sqrt{m^2 + \vec{p}^2}$).

Finally, we get (see eqs. (162) and (164)):

$$\hat{\psi}(\vec{x}, t) = e^{i\hat{P}\vec{x}} \hat{\psi}(0) e^{-i\hat{P}\vec{x}} \Leftrightarrow \hat{\psi}(\vec{x}) = e^{i\hat{P}\vec{x}} \hat{\psi}(0) e^{-i\hat{P}\vec{x}} \quad (166)$$

Similarly, one can show that

$$\hat{\pi}(\vec{x}) = e^{i\hat{P}\vec{x}} \hat{\pi}(0) e^{-i\hat{P}\vec{x}}$$

At equal times, operators $\hat{\psi}(\vec{x})$ and $\hat{\pi}(\vec{x})$ commute according to canonical commutation relations

$$[\hat{\psi}(\vec{x}, t), \hat{\pi}(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y})$$

$$[\hat{\psi}(\vec{x}, t), \hat{\psi}(\vec{y}, t)] = [\hat{\pi}(\vec{x}, t), \hat{\pi}(\vec{y}, t)] = 0 \quad (167)$$

"equal-time
commutators"

Proof:

$$[\hat{\psi}(\vec{x}, t), \hat{\pi}(\vec{y}, t)] = e^{i\hat{H}t} \hat{\psi}(\vec{x}) e^{-i\hat{H}t} \cancel{e^{i\hat{H}t}} \hat{\pi}(\vec{y}) e^{-i\hat{H}t} - (\psi(\vec{x}) \Leftrightarrow \pi(\vec{y})) -$$

$$= e^{i\hat{H}t} [\hat{\psi}(\vec{x}), \hat{\pi}(\vec{y})] e^{-i\hat{H}t} = e^{i\hat{H}t} i\delta(\vec{x} - \vec{y}) e^{-i\hat{H}t} = i\delta(\vec{x} - \vec{y})$$

Similarly, one can prove second line in eq. (167)

Wave function of a free scalar particle

$$|p\rangle \equiv \sqrt{2E_p} a_p^\dagger |0\rangle \quad - \text{one-particle state (after Peskin)}$$

$$|\tilde{p}\rangle \equiv a_p^\dagger |0\rangle \quad - \text{one-particle state from Bjorken textbook (different normalization)}$$

Definition

$$f_p(x) = \langle 0 | \hat{\psi}(x) | \tilde{p} \rangle \leftarrow \text{wave function from AQM course}$$

Explicit form:

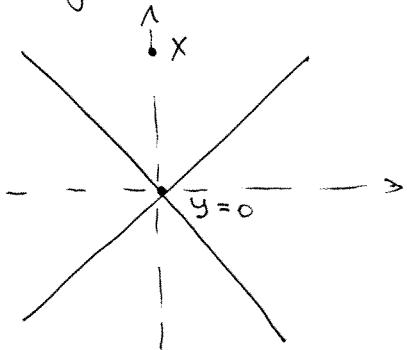
$$\begin{aligned}
 \hat{f}_p(x) &= \langle 0 | \hat{\psi}(x) \hat{a}_p^+ | 0 \rangle = \langle 0 | \int \frac{dp'}{\sqrt{2E_p}} (a_p' e^{-ip'x} + a_p' e^{ip'x}) \hat{a}_p^+ | 0 \rangle \\
 &= \int \frac{dp'}{\sqrt{2E_p}} e^{-ip'x} \langle 0 | [\hat{a}_p', \hat{a}_p^+] | 0 \rangle = \int \frac{dp'}{\sqrt{2E_p}} e^{-ip'x} (2\pi)^3 \delta(\vec{p} - \vec{p}') \\
 \Rightarrow f_p(x) &= \frac{e^{-ipx}}{\sqrt{2E_p}} \quad \text{- same expression as in AQM course (plane wave)} \tag{168}
 \end{aligned}$$

Propagation amplitude

From AQM course:

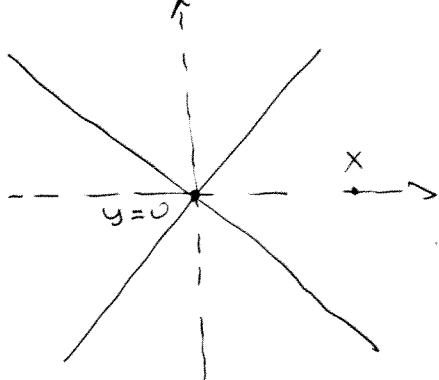
$$\begin{aligned}
 D(x-y) \equiv K_0(x-y) &= \sum_n \psi_n(x) \psi_n^*(y) \rightarrow \int dp \frac{e^{-ipx}}{\sqrt{2E_p}} \frac{e^{ipy}}{\sqrt{2E_p}} \Big|_{p_0=E_p} \\
 &= \int dp \frac{e^{-ip(x-y)}}{2E_p} \Big|_{p_0=E_p} \quad D(x-y) = \langle 0 | \psi(x) \psi(y) | 0 \rangle \tag{169}
 \end{aligned}$$

Large-distance behavior of $D(x-y)$



At time-like intervals:
find a frame where $x-y = (t, \vec{0})$

$$\begin{aligned}
 D(x-y) &= \frac{1}{4\pi^2} \int_0^\infty dp \frac{p^2}{\sqrt{m^2+p^2}} e^{-it\sqrt{m^2+p^2}} = \\
 &= \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2-m^2} e^{-iEt} = \boxed{E = \xi + m} \\
 &= \frac{1}{4\pi^2} e^{-imt} \int_0^\infty d\xi e^{-i\xi t} \sqrt{\xi^2+2m\xi} \rightarrow \\
 &\quad t \rightarrow \infty \quad \xi \approx 0 \\
 \Rightarrow & \frac{e^{-imt}}{4\pi^2} \frac{\sqrt{2m}}{(it)^{3/2}} \Gamma(\frac{3}{2}) = \frac{\sqrt{m}}{2(2\pi i t)^{3/2}} e^{-imt} \tag{170}
 \end{aligned}$$



At space-like intervals:
find a frame where $x-y = 0, \vec{r}$

$$D(x-y) = \int \frac{dp}{2E_p} e^{i\vec{p}\vec{r}} = \int \frac{p^2 dp d\Omega}{(2\pi)^3 2E_p} e^{ipr \cos\theta} =$$

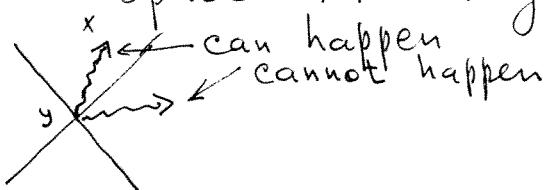
$$\begin{aligned}
&= \frac{1}{4\pi^2} \int_0^\infty \frac{p^2 dp}{2E_p} \int_{-\pi}^\pi \sin\theta d\theta e^{iprcos\theta} = \frac{1}{4\pi^2} \int_0^\infty \frac{p^2 dp}{2E_p} \int_{-1}^1 dt e^{iprt} = \\
&= \frac{1}{4\pi^2} \int_0^\infty \frac{p^2 dp}{2\sqrt{m^2 + p^2}} \frac{e^{ipr} - e^{-ipr}}{ipr} = -\frac{i}{8\pi^2 r} \int_{-\infty}^\infty dp \frac{pe^{ipr}}{\sqrt{m^2 + p^2}} = \frac{-i}{8\pi^2 r} \int_c^\infty dp \frac{pe^{ipr}}{\sqrt{m^2 + p^2}}
\end{aligned}$$

push contour

$$\begin{aligned}
&= \frac{1}{4\pi^2 r} \int_m^\infty dg \frac{ge^{-rg}}{\sqrt{g^2 - m^2}} = \\
&= \frac{e^{-mr}}{4\pi^2 r} \int_0^\infty dh \frac{e^{-\lambda r}}{\sqrt{\lambda^2 + 2\lambda m}} \xrightarrow{\lambda \sim r} \\
&\xrightarrow{r \rightarrow \infty} \frac{\sqrt{m} e^{-mr}}{2(2\pi r)^{3/2}} \quad (171)
\end{aligned}$$

Causality

Causality \Leftrightarrow there should be no signal going into the space-like region



In other words, measurement performed at x should not affect measurement performed at y if $(x-y)^2 < 0$.

"Elementary measurement" is $[\psi(x), \psi(y)]$
 in a field theory

Thus, causality $\Leftrightarrow [\psi(x), \psi(y)] \neq 0$ for $(x-y)^2 < 0$

Check:

$$\begin{aligned}
[\psi(x), \psi(y)] &= \int \frac{d^3 p}{\sqrt{2E_p}} \frac{d^3 p'}{\sqrt{2E'_p}} \left[[a_p^+, a_{p'}^+] e^{-ipx + ip'y} + [a_p^+, a_{p'}^-] e^{ipx - ip'y} \right] \Big|_{p_0=E_p} = \\
&= \int \frac{d^3 p}{2E_p} (e^{-ip(x-y)} - e^{ip(x-y)}) \Big|_{p_0=E_p} = D(x-y) - D(y-x) \quad (172)
\end{aligned}$$

If $(x-y)^2 < 0$, this is 0 because $D(x-y) = D(y-x)$

Proof:

For $(x-y)^2 < 0$ there exists a frame where $x-y = 0, \vec{r}$

Then $D(x-y) = \frac{1}{4\pi^2} \int_m^\infty dg e^{-rg} (g^2 - m^2)^{1/2} \Rightarrow D(x-y) = D(y-x) = f(\vec{r})$

If $(x-y)^2 > 0 \quad [\varphi(x), \varphi(y)] \neq 0$

for example, at $x-y = (t, \vec{0})$ and $t \rightarrow \infty \quad D(x-y) \approx e^{-i\omega t}$
 $D(y-x) \approx e^{i\omega t}$ so $D(x-y) - D(y-x) \sim e^{-i\omega t} - e^{i\omega t} \neq 0$.

Feynman propagator

From AQM : $D_F(x-y) \equiv G_0(x-y) = \Theta(x_0-y_0) D(x-y) + \Theta(y_0-x_0) D(y-x)$ (173)

In terms of operators

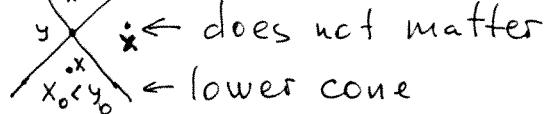
$$D_F(x-y) = \Theta(x_0-y_0) \langle 0 | \varphi(x) \varphi(y) | 0 \rangle + \Theta(y_0-x_0) \langle 0 | \varphi(y) \varphi(x) | 0 \rangle$$

Definition : $T\{\varphi(x) \varphi(y)\} \equiv \Theta(x_0-y_0) \varphi(x) \varphi(y) + \Theta(y_0-x_0) \varphi(y) \varphi(x)$ (174)

$$\Rightarrow D_F(x-y) = \langle 0 | T\{\varphi(x) \varphi(y)\} | 0 \rangle$$

"Feynman propagator"
or "Feynman Green function"

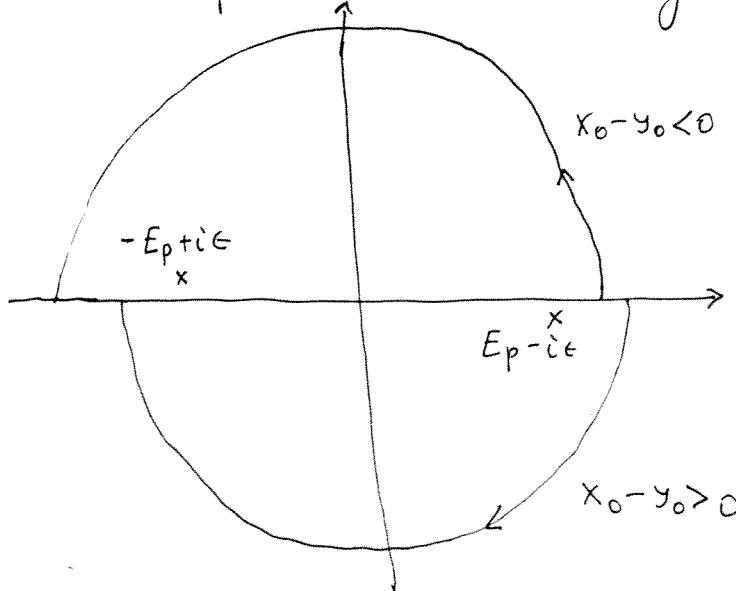
since $[\varphi(x), \varphi(y)] = 0$ at $(x-y)^2 < 0$, this definition is relativistic invariant:



Explicit form of Feynman propagator (reminder)

$$D_F(x-y) = \lim_{\epsilon \rightarrow 0} \int \frac{d^4 p}{i} \frac{e^{-ip(x-y)}}{m^2 - p^2 - i\epsilon} \quad (175)$$

Proof : perform the integration over p_0



(p_0)

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int \frac{d^3 p}{i} e^{i\vec{p}(\vec{x}-\vec{y})} \cdot \int \frac{d p_0}{i} \frac{(-1) e^{-i p_0(x_0-y_0)}}{(p_0 - E_p + i\epsilon)(p_0 + E_p - i\epsilon)} = \\ & = \Theta(x_0-y_0) \int \frac{d^3 p}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y}) - iE_p(x_0-y_0)} \\ & + \Theta(y_0-x_0) \int \frac{d^3 p}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y}) + iE_p(x_0-y_0)} \\ & = (\text{charge } \vec{p} \leftrightarrow -\vec{p}) = \end{aligned}$$

$$= \Theta(x_0 - y_0) \underbrace{\int \frac{d^3 p}{2E_p} e^{-ip(x-y)} \Big|_{p_0 = E_p}}_{D(x-y)} + \Theta(y_0 - x_0) \underbrace{\int \frac{d^3 p}{2E_p} e^{ip(x-y)} \Big|_{p_0 = E_p}}_{D(y-x)} = D_F(x-y)$$

Recall notation from AQM course :

$$G_0(p) = \frac{1}{m^2 - p^2 - i\epsilon} \quad (\lim_{\epsilon \rightarrow 0} \text{ implied}) \quad (176)$$

(From now on, $\lim_{\epsilon \rightarrow 0}$ is implied)

Mathematically, the Feynman propagator (175) is a Green function of the Klein-Gordon equation

$$\begin{aligned} (\partial_x^2 + m^2) D_F(x-y) &= \int \frac{d^4 p}{i} \frac{1}{m^2 - p^2 - i\epsilon} \left(\frac{\partial}{\partial p_\mu} \frac{\partial}{\partial x^\mu} + m^2 \right) e^{-ip(x-y)} = \\ &= \int \frac{d^4 p}{i} \frac{1}{m^2 - p^2 - i\epsilon} (-p^2 + m^2) e^{-ip(x-y)} = -i \delta^{(4)}(x-y) \end{aligned} \quad (177)$$

Retarded and advanced Green functions

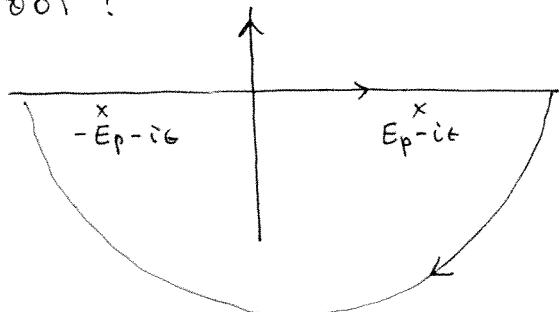
$$D_R(x-y) \equiv \Theta(x_0 - y_0) \langle 0 | [\varphi(x), \varphi(y)] | 0 \rangle - \text{retarded Green function} \quad (178)$$

$$D_A(x-y) \equiv \Theta(y_0 - x_0) \langle 0 | [\varphi(y), \varphi(x)] | 0 \rangle - \text{advanced Green function} \quad (179)$$

Explicit form

$$D_R(x-y) = \int \frac{d^4 p}{i} \frac{e^{-ip(x-y)}}{m^2 - p^2 - i\epsilon p_0} \quad (180)$$

Proof :



$$\begin{aligned} &\int \frac{d^3 p}{2E_p} e^{+i\vec{p}(\vec{x}-\vec{y})} \int d^4 p_0 \frac{i e^{-ip_0(x_0-y_0)}}{(p_0 - E_p + i\epsilon)(p_0 + E_p + i\epsilon)} \\ &= \Theta(x_0 - y_0) \int \frac{d^3 p}{2E_p} e^{-iE_p(x_0-y_0) + i\vec{p}(\vec{x}-\vec{y})} \\ &- \Theta(x_0 - y_0) \int \frac{d^3 p}{2E_p} e^{+iE_p(x_0-y_0) + i\vec{p}(\vec{x}-\vec{y})} \\ &= \Theta(x_0 - y_0) (D(x-y) - D(y-x)) = (178) \end{aligned}$$

Similarly,

$$D_A(x-y) = \int \frac{d^4 p}{i} \frac{e^{-ip(x-y)}}{m^2 - p^2 + i\epsilon p_0} = \Theta(y_0 - x_0) (D(y-x) - D(x-y)) = (179)$$